Math 6345- AODE's

So $e^{At} = I + At + \frac{At^2}{2!} + \cdots$

but this def^n is hard to use

Putzer's Algorithm

Let A be a $n \times n$ matrix with eigenvalues

$$\lambda_1, \lambda_2, \ldots \lambda_n$$

then $$e^{At} = \sum_{j=0}^{n-1} r_{j+1}(t) P_j$$

where $P_0 = I$, $P_j = \prod_{k=1}^{j} (A - \lambda_k I) \quad j = 1, 2 \ldots n$

& $r_1, r_2 \ldots r_n$ sol^n's of

$$\left.\begin{array}{l} \dot{r}_1 = \lambda_1 r_1, \quad \dot{r}_j = r_{j-1} + \lambda_j r_j \\ r_1(0) = 1, \quad r_j(0) = 0 \end{array}\right\} j = 2, 3 \ldots n$$

Proof $n=2$

So $$e^{At} = r_1 P_0 + r_2 P_1$$
$$= r_1 I + r_2 (A - \lambda_1 I)$$

let this be $= F$. we show $\dot{F} = AF$

if $$F = r_1 I + r_2 (A - \lambda_1 I)$$
$$\dot{F} = \dot{r}_1 I + \dot{r}_2 (A - \lambda_1 I)$$
$$= \lambda_1 r_1 I + (r_1 + \lambda_2 r_2)(A - \lambda_1 I)$$
$$= \cancel{\lambda_1 r_1 I} + r_1 (A - \cancel{\lambda_1 I}) + \lambda_2 r_2 (A - \lambda_1 I)$$
$$= r_1 A + \lambda_2 r_2 A - \lambda_1 \lambda_2 r_2 I$$

Now A satisfies $(A - \lambda_1 I)(A - \lambda_2 I) = 0$
$$\Rightarrow \quad A^2 - \lambda_1 A - \lambda_2 A + \lambda_1 \lambda_2 I = 0$$
$$A^2 - \lambda_1 A = \lambda_2 A - \lambda_1 \lambda_2 I$$

$$\dot{F} = r_1 A + (A^2 - \lambda_1 A) r_2 = A\left(r_1 I + r_2 (A - \lambda_1 I)\right)$$
$$= AF.$$

& $F(0) = I$

$\square$

Ex1 $\dot{\bar{x}} = \begin{pmatrix}1&0\\0&2\end{pmatrix}\bar{x}$

$$\begin{vmatrix}\lambda-1 & 0\\ 0 & \lambda-2\end{vmatrix}=0 \qquad \lambda=1,2$$

$$e^{At} = \begin{pmatrix} e^t & 0\\ 0 & e^{2t}\end{pmatrix}$$

using Putzer's Alg.

$\dot r_1 = r_1 \qquad r_1(0)=1 \quad \Rightarrow \quad r_1 = ce^t \qquad r_1(0)=1$

$\dot r_2 = r_1+2r_2 \qquad r_2(0)=0 \qquad \boxed{r_1=e^t}$

$\dot r_2 - 2r_2 = e^t \qquad \mu = e^{-2t}$

$\frac{d}{dt}(e^{-2t}r_2) = e^{-t} \Rightarrow e^{-2t}r_2 = -e^{-t}+c$

$r_2 = ce^{2t} - e^t \qquad r_2(0)=0 \Rightarrow c_1=1$

so $\boxed{r_2 = e^{2t}-e^t}$

$P_0 = \begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad P_1 = \begin{pmatrix}1&0\\0&2\end{pmatrix} - 1\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\0&1\end{pmatrix}$

$$e^{At} = r_1 P_0 + r_2 (A - \lambda_1 I)$$

$$\doteq e^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (e^{2t} - e^t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & 0 \\ 0 & e^t + e^{2t} - e^t \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

<u>Ex2</u> $A = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$

$\begin{vmatrix} \lambda - 4 & 3 \\ -3 & \lambda + 2 \end{vmatrix} = 0$ $\quad (\lambda-4)(\lambda+2) + 9 = 0$

$\lambda^2 - 2\lambda - 8 + 9 = 0$

$\lambda^2 - 2\lambda + 1 = 0 \quad (\lambda - 1)^2 = 0$

$\lambda = 1, 1$

so $\dot{r}_1 = r_1 \quad r_1(0) = 1 \quad r_1 = e^t$

$\dot{r}_2 = r_1 + r_2 \quad r_2(0) = 0$

$\frac{dr_2}{dt} - r_2 = e^t \quad \frac{d}{dt} e^{-t} r_2 = 1 \quad e^{-t} r_2 = t + c_1$

$r_2 = (t + c_1) e^t \quad r_2(0) = 0 \Rightarrow c_1 = 0 \quad r_2 = t e^t$

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_1 = A - I = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix}$$

so

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{t} + \begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} t e^{t}$$

$$= \begin{pmatrix} (1+3t)e^{t} & -3te^{t} \\ 3te^{t} & (1-3t)e^{t} \end{pmatrix}$$

Ex. 3 $\quad A = \begin{pmatrix} 2 & -3 & -3 \\ 1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}$

so $|\lambda I - A| = 0 \Rightarrow \lambda = 0, -1, -1$

so how do we choose $\lambda_1, \lambda_2, \lambda_3$

Choice 1 $\quad \lambda_1 = 0, \quad \lambda_2 = -1, \quad \lambda_3 = -1$

so $\dot{r}_1 = 0 r_1 \Rightarrow r_1 = c_1 \quad r_1(0) = 1 \Rightarrow \boxed{r_1 = 1}$

$\dot{r}_2 = -r_2 + r_1 \qquad \dot{r}_2 = -r_2 + 1$

$\dot{r}_3 = -r_3 + r_2$

so $\frac{d}{dt} r_2 + r_2 = 1 \qquad \frac{d}{dt} e^t r_2 = e^t$

$\Rightarrow e^t r_2 = e^t + c_1 \qquad r_2(0) = 0 \Rightarrow c_1 = -1$

$$\boxed{r_2 = 1 - e^{-t}}$$

$\dot{r}_3 + r_3 = 1 - e^{-t} \qquad \frac{d}{dt}(e^t r_3) = e^t - 1$

$e^t r_3 = e^t - t + c_3$

$r_3(0) = 0 \qquad c_3 = -1$

$$\boxed{r_3 = 1 - te^{-t} - e^{-t}}$$

Now $P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P_1 = A - 0I = \begin{pmatrix} 2 & -3 & -3 \\ 1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}$

$$P_2 = (A - 0I)(A+I)$$

A

$$= \begin{pmatrix} 2 & -3 & -3 \\ 1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 3 & -3 & -3 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} = [0]$$

$$e^{At} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot 1 + \begin{pmatrix} 2 & -3 & -3 \\ 1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} (1 - e^{-t}) + [0] r_3$$

$$= \begin{pmatrix} 3 - 2e^{-t} & -3(1-e^{-t}) & -3(1-e^{-t}) \\ 1 - e^{-t} & -1 + 2e^{-t} & -(1-e^{-t}) \\ 1 - e^{-t} & -(1-e^{-t}) & -1 + 2e^{-t} \end{pmatrix}$$

Choice 2 $\quad \lambda_1 = -1, \quad \lambda_2 = -1, \quad \lambda_3 = 0$

$\dot{r}_1 = -r_1 \Rightarrow r_1 = e^{-t}$

$\dot{r}_2 = -r_2 + r_1 \Rightarrow \dot{r}_2 + r_2 = e^{-t} \qquad \frac{d}{dt} e^{t} r_2 = 1$

$e^{t} r_2 = t + c_1 \qquad r_2(0) = 0 \Rightarrow c_1 = 0 \qquad r_2 = t e^{-t}$

$$\dot{r}_3 = 0r_3 + r_2 = te^{-t} \Rightarrow r_3 = -(1+t)e^{-t} + c$$

$$r_3(0) = 0 \Rightarrow c_1 = 1 \qquad r_3 = 1 - (1+t)e^{-t}$$

so Now

$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P_1 = A - \lambda_1 I = A + I = \begin{pmatrix} 3 & -3 & -3 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$P_2 = (A-\lambda_1 I)(A-\lambda_2 I) = (A+I)^2 = \begin{pmatrix} 3 & -3 & -3 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 3 & -3 & -3 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \left(te^{-t} + 1 - (1+t)e^{-t} \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 3 & -3 & -3 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} (1 - e^{-t})$$

$$= \begin{pmatrix} 3-2e^{-t} & -3(1-e^{-t}) & -3(1-e^{-t}) \\ 1-e^{-t} & -1+2e^{-t} & -(1-e^{-t}) \\ 1-e^{-t} & -(1-e^{-t}) & -1+2e^{-t} \end{pmatrix} \text{ Same}$$