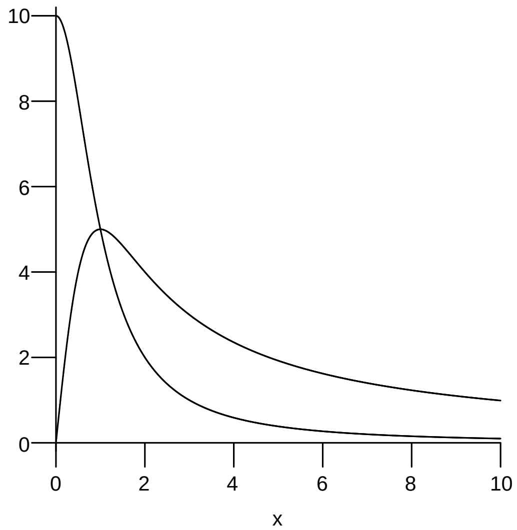
Math 6345 Advanced ODEs
An example of a Trapping Region

We consider

$$\dot{x} = -x + ay + x^2 y, \quad \dot{y} = b - ay - x^2 y, \quad a, b > 0,$$

where $x, y > 0$. We first consider the nullclines, *i.e.* the curves where $\dot{x} = 0$ and $\dot{y} = 0$. Theses are given by

$$\begin{aligned} y &= \frac{x}{a + x^2} \quad (\dot{x} = 0) \quad \text{trajectories are vertical} \\ y &= \frac{b}{a + x^2} \quad (\dot{y} = 0) \quad \text{trajectories are horizontal} \end{aligned}$$



We see that the first quadrant is divided into four region depending on whether $y \lesseqgtr \frac{x}{a+x^2}$ and $y \lesseqgtr \frac{b}{a+x^2}$ (see the picture above). In each region we will determine the flow.

Region 1 $y < \frac{x}{a+x^2} \quad y < \frac{b}{a+x^2}$,
Since

$$\begin{aligned} y < \frac{x}{a + x^2} \quad &\Rightarrow \quad -x + (a + x^2)y < 0 \quad \Rightarrow \quad \dot{x} < 0, \\ y < \frac{b}{a + x^2} \quad &\Rightarrow \quad b - (a + x^2)y > 0 \quad \Rightarrow \quad \dot{y} > 0. \end{aligned}$$

Region 2 $y > \frac{x}{a+x^2} \quad y < \frac{b}{a+x^2}$,
Since

$$\begin{aligned} y > \frac{x}{a + x^2} \quad &\Rightarrow \quad -x + (a + x^2)y > 0 \quad \Rightarrow \quad \dot{x} > 0, \\ y < \frac{b}{a + x^2} \quad &\Rightarrow \quad b - (a + x^2)y > 0 \quad \Rightarrow \quad \dot{y} > 0. \end{aligned}$$

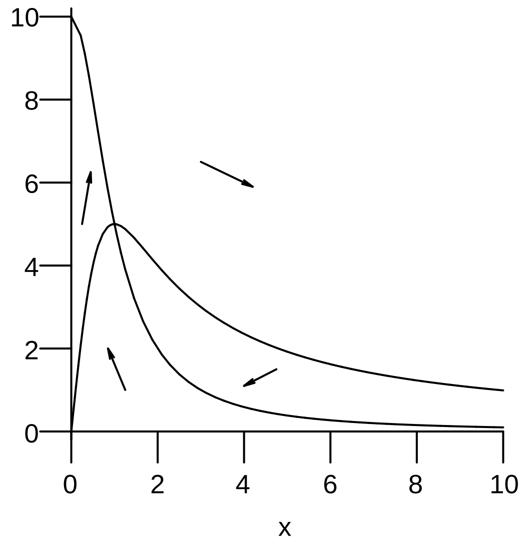
Region 3 $y > \frac{x}{a+x^2} \quad y > \frac{b}{a+x^2}$,
Since

$$y > \frac{x}{a+x^2} \quad \Rightarrow \quad -x+(a+x^2)y \; > 0 \quad \Rightarrow \quad \dot{x} > 0,$$
$$y > \frac{b}{a+x^2} \quad \Rightarrow \quad b-(a+x^2)y \; < 0 \quad \Rightarrow \quad \dot{y} < 0.$$

Region 4 $y < \frac{x}{a+x^2} \quad y > \frac{b}{a+x^2}$,
Since

$$y < \frac{x}{a+x^2} \quad \Rightarrow \quad -x+(a+x^2)y \; < 0 \quad \Rightarrow \quad \dot{x} < 0,$$
$$y > \frac{b}{a+x^2} \quad \Rightarrow \quad b-(a+x^2)y \; < 0 \quad \Rightarrow \quad \dot{y} < 0.$$

This is depicted in the picture below.



Next we consider the following region: $y = b/a$ for $0 < x < b$, a line from the point $(b, b/a)$ to the curve $y = \frac{x}{a+x^2}$ with a slope of -1, a vertical line down to the x - axis and then the x- and y - axis (see the figure). We determine the flow on each of these boundaries.

Boundary 1 x - axis
On this boundary, $y = 0$ so

$$\dot{x} = -x < 0, \quad \dot{y} = b > 0,$$

Boundary 2 y - axis
On this boundary, $x = 0$ so

$$\dot{x} = ay > 0, \quad \dot{y} = b - ay > 0 \;\text{ since }\; y < b/a.$$

Boundary 3 $y = b/a$

On this boundary, $x = 0$ so

$$\dot{x} = \underbrace{-x + b}_{>0} + \underbrace{x^2 \frac{b}{a}}_{>0} > 0, \quad \dot{y} = -\frac{x^2 b}{x} < 0.$$

Boundary 4 Line with slope -1

On this boundary, we consider $\dot{y}/\dot{x}$

$$\frac{dy}{dx} = \frac{b - ay - x^2 y}{-x + ay + x^2 y} = -1 + \frac{b - x}{-x + ay + x^2 y} < -1.$$

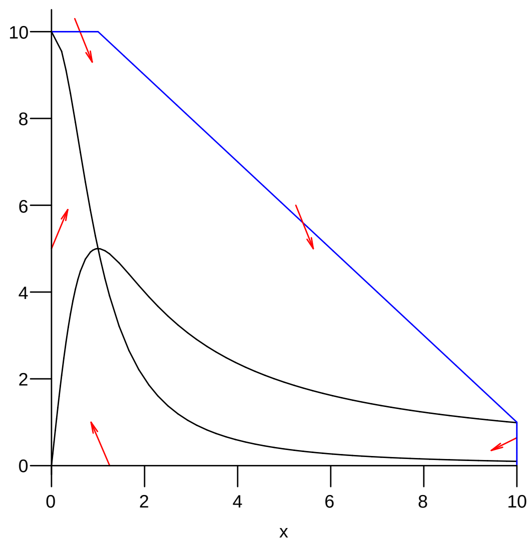
since on the line $-x + ay + x^2 y > 0$ and $x > b$.

Boundary 5 Vertical Line from $y = \frac{x}{a+x^2}$ to x-axis

On this boundary, we are between the two regions and from above

$$\dot{x} < 0 \quad \dot{y} < 0.$$

So the region is a trapping region (see below).



Next we consider the stability of the critical point $(b, \frac{b}{a+b^2})$. The linearized matrix is

$$\begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -a - x^2 \end{pmatrix}$$

and at the critical point

$$\begin{pmatrix} -1 + \frac{2b^2}{a+b^2} & a + b^2 \\ \frac{-2b^2}{a+b^2} & -a - b^2 \end{pmatrix}$$

The eigenvalues are found by solving

$$\lambda^2 + \left(a + b^2 + 1 - \frac{2b^2}{a+b^2}\right)\lambda + a - b^2 = 0.$$

In order to have nonzero real part (and unstable) we need

$$a + b^2 + 1 - \frac{2b^2}{a+b^2} < 0$$

Since all the flow is into the trapping region and the critical point in side the region is unstable meaning we can construct a little region around the critical point where the flow is outward, the by the Poincare-Bendixson theorem, there exist a periodic orbit.