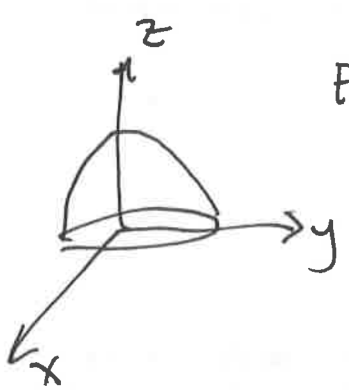
Math 2471 – Calc 3

Divergence Thm

Let V be a simple solid region and S the boundary surface with an outward normal $\hat{n}$. Let $\bar{F}$ be a vector field whose components have cont. derivatives.

then
$$\iint_S \bar{F}\cdot\hat{n}\,dS = \iiint_V \bar{\nabla}\cdot\bar{F}\,dV$$

Ex. 1 S: $z=16-x^2-y^2$ & $\bar{F}=\langle y, x, z\rangle$



First, the surface integrals

There are 2 surfaces

(i) paraboloid

(ii) disc when $z=0$

$S_1: \quad z = 16 - x^2 - y^2 = f$

so $\vec{n} = \langle -f_x, -f_y, 1 \rangle = \langle 2x, 2y, 1 \rangle$ it points outward ✓

so unit normal is

$$\hat{n} = \frac{\langle 2x, 2y, 1 \rangle}{\sqrt{1 + 4x^2 + 4y^2}}$$

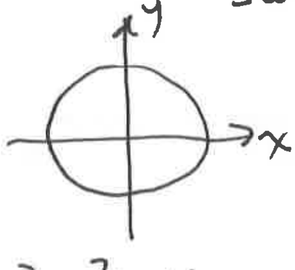
Next $\quad dS = \sqrt{1 + f_x^2 + f_y^2}\, dA = \sqrt{1 + 4x^2 + 4y^2}\, dA$

Now $\quad F \cdot \hat{n}\, dS = \langle y, x, z \rangle \cdot \dfrac{\langle 2x, 2y, 1 \rangle}{\cancel{\sqrt{1+4x^2+4y^2}}} \cancel{\sqrt{1+4x^2+4y^2}}\, dA$

$$= (2xy + 2xy + z)\, dA = (4xy + 16 - x^2 - y^2)\, dA$$

bring in surface

Since the region of integration is a circle of radius 4 switch to polar



$x^2 + y^2 = 16$

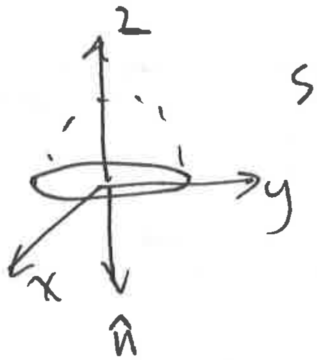
$$\int_{\theta=0}^{2\pi} \int_{r=0}^{4} (4r^2 \cos\theta \sin\theta + 16 - r^2)\, r\, dr\, d\theta$$

$$= 128\pi$$

S_2: <u>Bottom surface</u>

Here $z=0$ so $\bar{F} = \langle y, x, 0 \rangle$

& $\hat{n} = \langle 0, 0, -1 \rangle$



so $\bar{F} \cdot \hat{n} = \langle y, x, 0 \rangle \cdot \langle 0, 0, -1 \rangle$

$= 0$

So there is no flux through S_2

so overall flux

$$\iint_S \bar{F} \cdot \hat{n}\, ds = 128\pi$$

<u>PART II</u> $\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z) = 1$

$$\text{so } \iiint_V 1\, dv = \int_{\theta=0}^{2\pi} \int_{r=0}^{4} \int_{z=0}^{16-r^2} r\, dz\, dr\, d\theta =$$

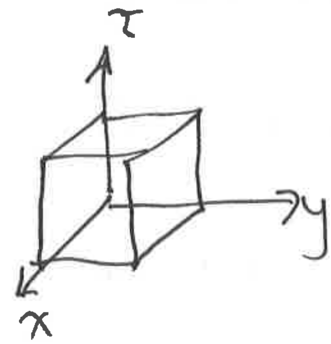
$$\int_0^{2\pi} \int_0^{4} r(16-r^2)\, dr\, d\theta = 128\pi$$

<u>Same!</u>

Ex 2 Verify the div Th^{m}

when V is the box bound by $x=0, x=1$; $y=0, y=1$; $z=0, z=1$

and $\bar{F} = \langle xy, 2y^2, -yz \rangle$

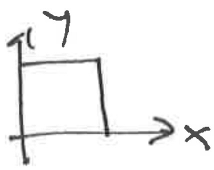


The box has 6 sides

S_1: top so $z=1$ $\quad \bar{F} = \langle xy, 2y^2, -y \rangle$

$\hat{n} = \langle 0, 0, 1 \rangle$

so $\bar{F} \cdot \hat{n} = -y$



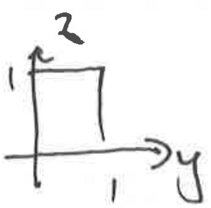
$$\int_0^1 \int_0^1 -y \, dy dx = -1/2$$

S_2: bottom so $z=0$ $\quad \bar{F} = \langle xy, 2y^2, 0 \rangle$ $\quad \hat{n} = \langle 0, 0, 1 \rangle$

so $\bar{F} \cdot \hat{n} = 0$ so no flux through S_2

S_3: front $x=1$, $\vec{F} = \langle y, 2y^2, -yz \rangle$ $\quad \hat{n} = \langle 1, 0, 0 \rangle$

so $\bar{F} \cdot \hat{n} = y$



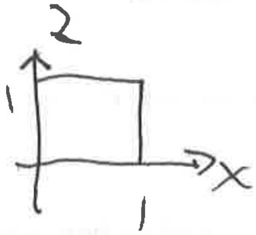
$$\int_0^1 \int_0^1 y \, dz dy = 1/2$$

S_4: Back $\quad x=0$ so $\quad \bar{F} = \langle 0, 2y^2, -yz \rangle$

$\hat{n} = \langle -1, 0, 0 \rangle$ so $\quad \bar{F}\cdot\hat{n} = 0 \quad$ no flux

S_5: right $\quad y=1$ so $\quad \bar{F} = \langle x, 2, -z \rangle$

$\hat{n} = \langle 0, 1, 0 \rangle$ & $\quad \bar{F}\cdot\hat{n} = 2$

so 

$$\int_0^1 \int_0^1 2\,dz\,dx = 2$$

S_6: left $\quad y=0 \qquad \bar{F} = \langle 0, 0, 0 \rangle$ so $\quad \bar{F}\cdot\hat{n} = 0$

Total Flux $= -\frac{1}{2} + 0 + \frac{1}{2} + 0 + 2 + 0 = 2$

Part 2 $\quad \bar{\nabla}\cdot\bar{F} = \frac{\partial}{\partial x} xy + \frac{\partial}{\partial y} 2y^2 + \frac{\partial}{\partial z}(-yz)$

$$= y + 4y - y = 4y$$

$$\int_0^1 \int_0^1 \int_0^1 4y\,dz\,dy\,dx = 2$$ Same!