Math 4381/6357 Nonlinear PDEs

In an introductory course in partial differential equations (PDEs) we learned how to solve a first order PDE of the form

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{1}$$

Using the chain rule $u_s = u_x x_s + u_y y_s$ and picking

$$x_s = a(x,y,u), \quad y_s = b(x,y,u), \tag{2}$$

the original PDE (1) becomes

$$u_s = c(x,y,u). \tag{3}$$

Here we have the equations (2) and (3) to solve for x, y and u. After integrating these, parameters r and s exist and after eliminating these, we arrive at the exact solution of the given PDE. The following examples demonstrate.

Example 1 Solve

$$xu_x + yu_y = 2u + y, \quad u(x,x^2) = 0. \tag{4}$$

If $u_s = u_x x_s + u_y y_s$, we pick

$$x_s = x, \quad y_s = y, \quad u_s = 2u + y. \tag{5}$$

We will solve this two ways. First, we will first solve the PDE to obtain the general solution and then impose the boundary conditions (BCs), and second, we will solve this subject to the BCs right away, *i.e.* the BCs will be passed into the new coordinate system (r,s).

<u>*Method One*</u> Solving the first two of (5) gives

$$x = a(r)e^s, \quad y = b(r)e^s, \tag{6}$$

where $a(r)$ and $b(r)$ are arbitrary functions of integration, and the remaining equation in (5) becomes

$$u_s = 2u + b(r)e^s. \tag{7}$$

Using the integrating factor $\mu = e^{-2s}$, we solve giving

$$u = -b(r)e^s + c(r)e^{2s} \tag{8}$$

where $c(r)$ is a third arbitrary function of integration. Eliminating s from (6) and (8) respectively, gives

$$\frac{y}{x} = \frac{b(r)}{a(r)} = A(r), \quad \frac{u+y}{y^2} = \frac{c(r)}{b^2(r)} = B(r). \tag{9}$$

Eliminating r then gives

$$\frac{u+y}{y^2} = B(A^{-1}(\frac{y}{x})), \tag{10}$$

or,

$$u = -y + y^2 f\left(\frac{y}{x}\right). \tag{11}$$

Now we impose the BC. On $y = x^2$, $u = 0$ so

$$0 = -x^2 + x^4 f\left(\frac{x^2}{x}\right) \tag{12}$$

or

$$f(x) = \frac{1}{x^2}. \tag{13}$$

Thus, we arrive at the final solution

$$u = -y + y^2\left(\frac{x}{y}\right)^2 = x^2 - y. \tag{14}$$

<u>*Method Two*</u> We will now re-do this problem but we will pass the BCs through to the (r, s) plane where we create a new boundary. As we have some freedom of choice we will pick $s = 0$ and connect the boundaries via $x = r$.

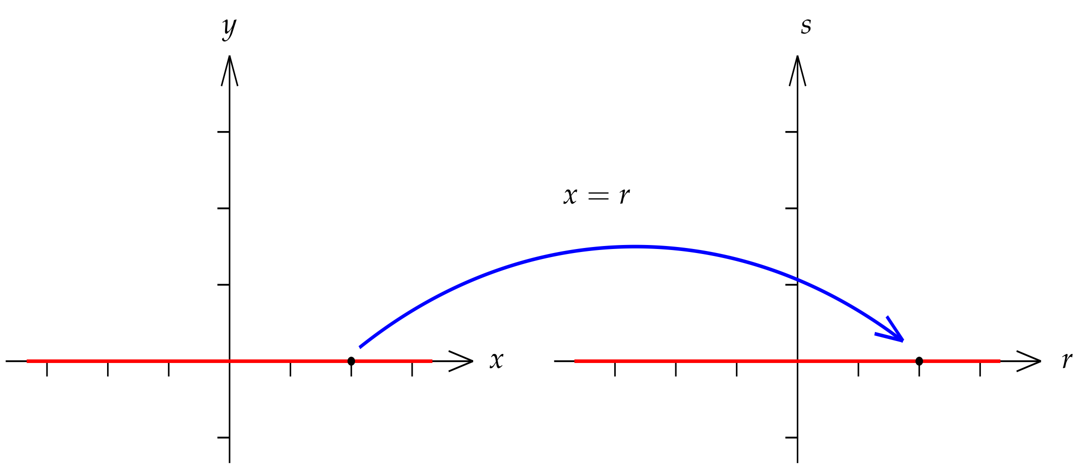

Figure 1. Change in the boundary from the (x, y) plane to the (r, s) plane.

When $s = 0$ we have the boundary conditions

$$x = r, \quad y = r^2, \quad u = 0. \tag{15}$$

Solving (6) again gives

$$x = a(r)e^s, \quad y = b(r)e^s, \tag{16}$$

however we impose the new BCs now giving $a(r) = r$ and $b(r) = r^2$ so

$$x = re^s, \quad y = r^2 e^s. \tag{17}$$

Solving (7) again gives

$$u = -r^2 e^s + c(r)e^{2s} \tag{18}$$

and imposing the third new BCs gives $c(r) = r^2$ and so

$$u = -r^2 e^s + r^2 e^{2s} \tag{19}$$

and eliminating r and s from (17) and (19) gives

$$u = x^2 - y, \tag{20}$$

which we saw previously.

Example 2 Solve

$$x^2 u_x - y^2 u_y = -y^2, \quad u(x, 1) = x \tag{21}$$

If $u_s = u_x x_s + u_y y_s$, we pick

$$x_s = x^2, \quad y_s = -y^2, \quad u_s = -y^2. \tag{22}$$

We create a new boundary in the (r, s) plane and we will pick $s = 0$ and connect the boundaries via $x = r$.

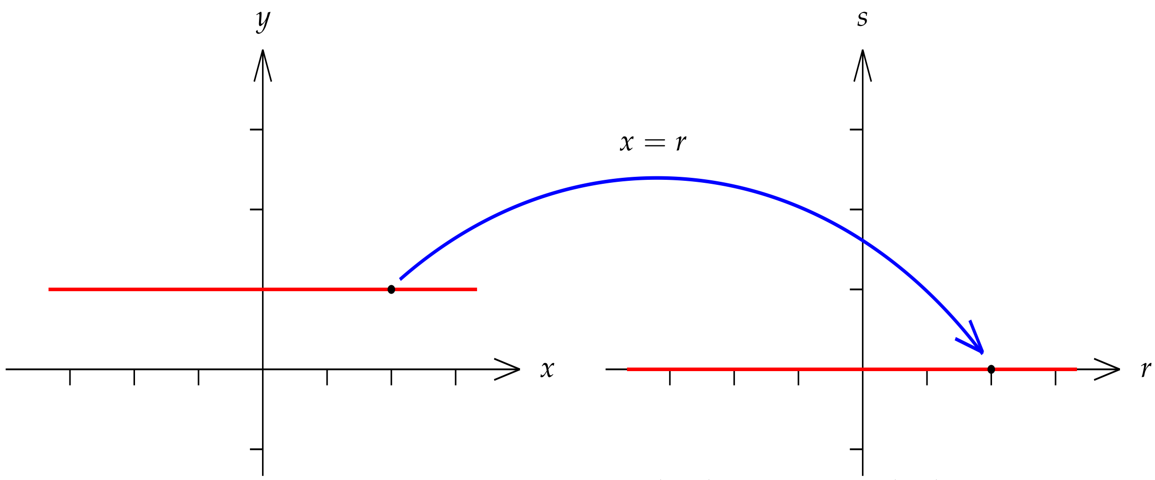

Figure 2. Change in the boundary from the (x, y) plane to the (r, s) plane.

The BCs we have on $s = 0$

$$x = r, \quad y = 1, \quad u = r. \tag{23}$$

Solving the first two of (22) gives

$$-\frac{1}{x} = s + a(r), \quad \frac{1}{y} = s + b(r). \tag{24}$$

Imposing the BCs for x and y gives $a(r) = -\frac{1}{r}$ and $b(r) = 1$ so

$$-\frac{1}{x} = s - \frac{1}{r}, \quad \frac{1}{y} = s + 1. \tag{25}$$

The remaining equation for u is then

$$u_s = -y^2 = -\frac{1}{(s+1)^2} \tag{26}$$

which easily integrates giving

$$u = \frac{1}{s+1} + c(r). \tag{27}$$

When $s = 0$, $u = r$ giving $c(r) = r - 1$ leading to

$$u = \frac{1}{s+1} + r - 1. \tag{28}$$

We solve (25) for r and s giving

$$r = \frac{xy}{x + y - xy}, \quad s = \frac{1}{y} - 1 \tag{29}$$

and use these in (28) giving our solution as

$$u = y - 1 + \frac{xy}{x + y - xy}. \tag{30}$$

Method of Characteristics

In order to solve

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{31}$$

we can bypass a lot of the previous steps and solve what is known as the *characteristic equations*

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}. \tag{32}$$

We demonstrate with example 1.

Solve

$$xu_x + yu_y = 2u + y, \quad u(x, x^2) = 0. \tag{33}$$

The characteristic equations are:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{2u+y}. \tag{34}$$

These we solve in pairs.

Pair 1:

$$\frac{dx}{x} = \frac{dy}{y} \quad \Rightarrow \quad \frac{y}{x} = c_1$$

Pair 2:

$$\frac{dy}{y} = \frac{du}{2u+y} \quad \Rightarrow \quad \frac{u}{y^2} + \frac{1}{y} = c_2.$$

The solution is then $c_2 = f(c_1)$ (or $c_1 = g(c_2)$) and

$$\frac{u}{y^2} + \frac{1}{y} = f\left(\frac{y}{x}\right).$$

Imposing the initial condition gives $f(x) = 1/x^2$ and we are led to the solution

$$u = x^2 - y \tag{35}$$

which was obtained in (14).

Example 3 Solve

$$(u+x)u_x + (u+y)u_y = u, \quad u(x,0) = x. \tag{36}$$

The characteristic equations are:

$$\frac{dx}{u+x} = \frac{dy}{u+y} = \frac{du}{u} \tag{37}$$

These we solve in pairs.

Pair 1:

$$\frac{dx}{u+x} = \frac{du}{u} \quad \Rightarrow \quad \frac{x}{u} - \ln u = c_1$$

Pair 2:

$$\frac{dy}{u+y} = \frac{du}{u} \quad \Rightarrow \quad \frac{y}{u} - \ln u = c_2.$$

The solution is then

$$\frac{x}{u} - \ln u = f\left(\frac{y}{u} - \ln u\right) \tag{38}$$

Imposing the initial condition $u(x,0) = x$ gives $f(-\ln x) = 1 - \ln x$ so $f(\lambda) = 1 + \lambda$ and we are lead to the solution

$$u = x - y. \tag{39}$$

A Nonlinear PDE

For fully nonlinear PDEs we need to introduce this new coordinate system again. If

$$F(x, y, u, u_x, u_y) = 0 \tag{40}$$

then we need to solve

$$x_s = F_p, \quad y_s = F_q, \quad u_s = pF_p + qF_q, \quad p_s = -F_x - pF_u, \quad q_s = -F_y - qF_u \tag{41}$$

where $p = u_x$ and $q = u_y$. The following example illustrates.

Example 4 Solve

$$u_x^2 + u_y = 2y + 1, \quad u(x,x) = x^2 + x \tag{42}$$

First we identify F (where $p = u_x$ and $q = u_y$) as

$$F = p^2 + q - 2y - 1$$

so

$$F_x = 0, \quad F_y = -2, \quad F_u = 0, \quad F_p = p^2, \quad F_q = 1.$$

The characteristic equations are

$$x_s = F_p = 2p, \tag{43a}$$

$$y_s = F_q = 1, \tag{43b}$$

$$u_s = p\,F_p + q\,F_q = 2p^2 + q, \tag{43c}$$

$$p_s = -F_x - p\,F_u = 0, \tag{43d}$$

$$q_s = -F_y - q\,F_u = 2. \tag{43e}$$

Next, we create new BCs in the (r, s) coordinate system. If we let $s = 0$ be the new boundary and connect the two boundaries ($y = x$ to $s = 0$), we will let $x = r$. Thus, we have the following. On $s = 0$

$$x = r, \quad y = r, \quad u = r^2 + r. \tag{44}$$

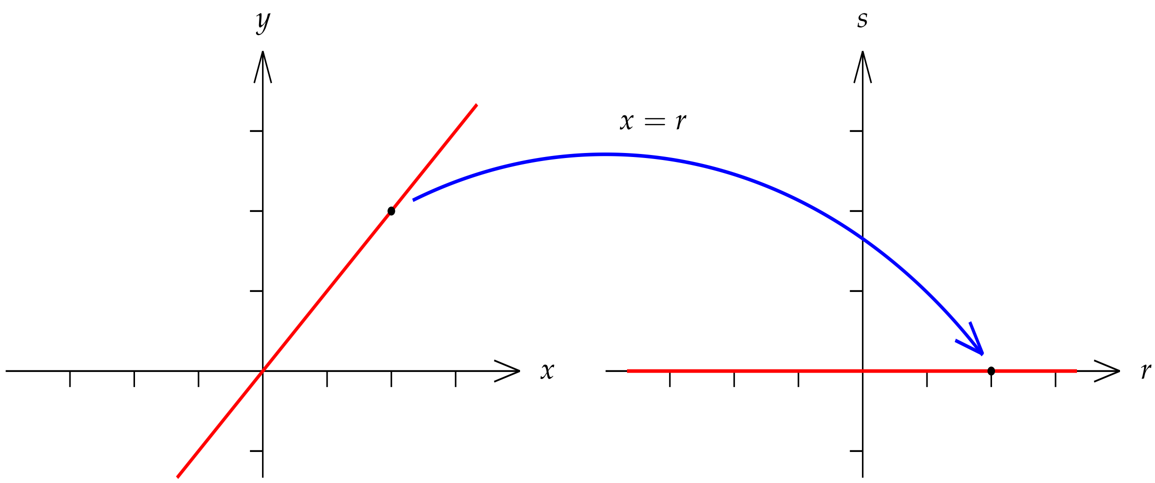

Figure 3. Change in the boundary from the (x, y) plane to the (r, s) plane.

As the system (43) has equations for p and q, we will need two more boundary conditions. To find these, we will use the existing boundary condition and the PDE itself. If we differentiate the giving boundary condition we obtain $u_x(x, x) + u_y(x, x) = 2x + 1$ (chain rule). This, with the given PDE gives

$$\begin{aligned} p^2 + q - 2r - 1 &= 0, \\ p + q - 2r - 1 &= 0, \end{aligned}$$

from which we deduce $p^2 - p = 0$ or $p = 0$ and $p = 1$ (two cases) and each, in turn, gives $q = 2r + 1$ and $q = 2r$.

Case 1

We to solve (43) subject to the BC

$$x = r, \quad y = r, \quad u = r^2 + r, \quad p = 0, \quad q = 2r + 1, \quad \text{on} \;\; s = 0 \tag{45}$$

(1) $p_s = 0$ solves giving $p = a(r)$. On $s = 0$ $p = 0$ so $a(r) = 0$ giving $p = 0$.

(2) $q_s = 2$ solves giving $q = 2s + b(r)$. On $s = 0$ $q = 2r + 1$ so $b(r) = 2r + 1$ giving $q = 2s + 2r + 1$.

(3) $x_s = 2p = 0$ solves giving $x = c(r)$. On $s = 0$ $x = r$ so $c(r) = r$ giving $x = r$.

(4) $y_s = 1$ solves giving $y = s + d(r)$. On $s = 0$ $y = r$ so $d(r) = r$ giving $y = s + r$.

(5) $u_s = 2p^2 + q = 2s + 2r + 1$ solves giving $q = s^2 + 2rs + s + e(r)$. On $s = 0$ $u = r^2 + r$ so $e(r) = r^2 + r$ giving $u = s^2 + 2rs + s + r^2 + r$.

At this point, we have the solution parametrically as

$$x = r, \quad y = s + r, \quad u = s^2 + 2rs + s + r^2 + r. \tag{46}$$

Eliminating r and s from x, y and u in (46) gives

$$u = y^2 + y.$$

Case 2

We to solve (43) subject to the BC

$$x = r, \quad y = r, \quad u = r^2 + r, \quad p = 1, \quad q = 2r, \quad \text{on} \;\; s = 0 \tag{47}$$

(1) $p_s = 0$ solves giving $p = a(r)$. On $s = 0$ $p = 1$ so $a(r) = 1$ giving $p = 1$.

(2) $q_s = 2$ solves giving $q = 2s + b(r)$. On $s = 0$ $q = 2r$ so $b(r) = 2r$ giving $q = 2s + 2r$.

(3) $x_s = 2p = 2$ solves giving $x = 2s + c(r)$. On $s = 0$ $x = r$ so $c(r) = r$ giving $x = 2s + r$.

(4) $y_s = 1$ solves giving $y = s + d(r)$. On $s = 0$ $y = r$ so $d(r) = r$ giving $y = s + r$.

(5) $u_s = 2p^2 + q = 2s + 2r + 2$ solves giving $q = s^2 + 2rs + 2s + e(r)$. On $s = 0$ $u = r^2 + r$ so $e(r) = r^2 + r$ giving $u = s^2 + 2rs + 2s + r^2 + r$.

At this point, we have the solution parametrically as

$$x = 2s + r, \quad y = s + r, \quad u = s^2 + 2rs + s + r^2 + r. \tag{48}$$

Eliminating r and s from x, y and u in (48) gives

$$u = y^2 + x.$$

Food for Thought

In the previous example, we needed to differentiate the boundary condition to find p and q on the boundary. Suppose we also differentiated the given PDE with respect to x and y, would any of these be useful? For example

$$2u_x u_{xx} + u_{xy} = 0, \quad 2u_x u_{xy} + u_{yy} = 2. \tag{49}$$

First opinion suggest no and we have increased the order to second order. However, if we let $v = u_x$, the first PDE becomes

$$2vv_x + v_y = 0 \tag{50}$$

and since $p = 0$ and $p = 1$ on the boundary, we obtain the conditions

$$v(x, x) = 0, \quad v(x, x) = 1 \tag{51}$$

thus, we have two quasilinear PDEs for v. Would these be easier to solve than the characteristic method and more importantly, would they lead to the same exact solutions?