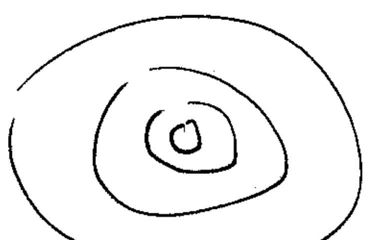
Math 1496 - Calc I

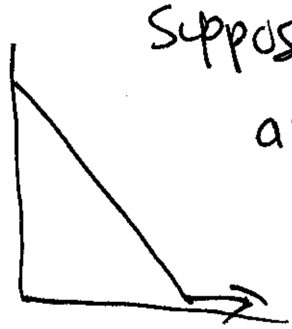
Related Rates

As the name suggests, we are going to relate 2 rates of change

Consider, for example, we drop a stone onto a pond. We will see a circular wave emerge. If we knew the rate at which the radius was changing could we find the rate at which the area is changing



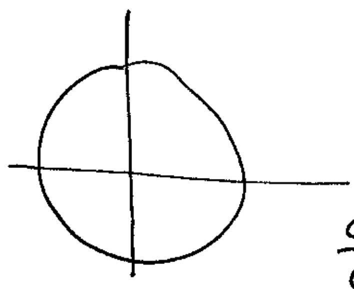
Suppose we have a ladder (10ft long) resting on the wall



Suppose we pull the base of the ladder at a rate of say 1ft/min how fast is the tip of the ladder falling when the foot is 6 ft away from the wall

These are examples of related rate.

Ex! If the radius of the circle is changing at 2"/sec how fast is the area changing when the circle is 12" radius



$A = \pi r^2$ ← we need this formula

$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2)\frac{dr}{dt}$ chain rule

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

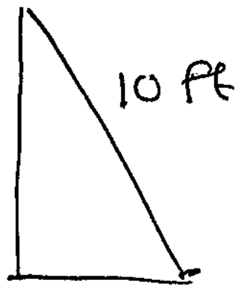
what do we know

(1) $\frac{dr}{dt} = 2''/sec$

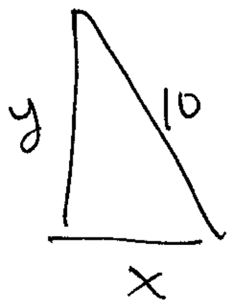
we want

$\frac{dA}{dt}$ at $r = 12''$ so $\frac{dA}{dt} = 2\pi(12)(2) = 48 \frac{\text{inch}^2}{\text{sec}}$

Ex 2



The ladder is being pulled away from the wall. First we introduce some variable



what do we know?

$$\frac{dx}{dt} = 1 \text{ ft/min}$$

what do we want to know

$\frac{dy}{dt}$ when $x = 6$

Relate variables

$$x^2 + y^2 = 10^2$$

~~at~~ we need y

$$6^2 + y^2 = 10^2$$
$$y = 8$$

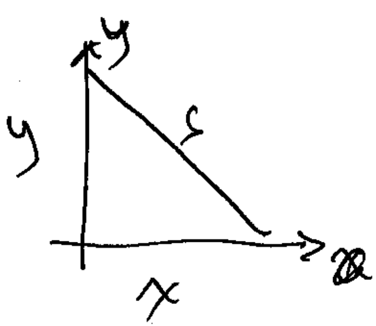
Relate variables

$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = 0 \qquad \frac{dy}{dt} = -\frac{x\frac{dx}{dt}}{y}$$

So $\frac{dy}{dt} = \frac{-6(1)}{8} = \frac{-3}{4} \text{ ft/min}$

ex 3 Two cars leave a city (same place) at 12 noon. Car A is travelling north at 50 mph, car B travelling east at 60. How fast is the distance between them changing after 1 hr?

(1) Picture & Label



we know $\frac{dx}{dt} = 60$ mph $\frac{dy}{dt} = 50$ mph

we want $\frac{ds}{dt}$ after 1 hr. BTW $x = 60$, $y = 50$

Relate variables $x^2 + y^2 = s^2$

Relate rates $\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}s\frac{ds}{dt}$

$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{s} = \frac{(60)(60) + (50)(50)}{\sqrt{60^2 + 50^2}} = 10\sqrt{61} \text{ mph}$$

$= 78.1$ mph

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Related Rates

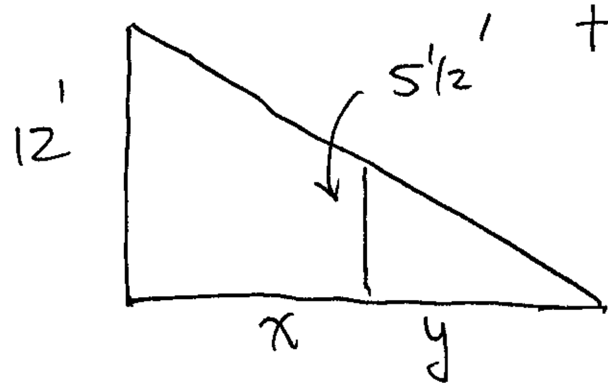
We consider 2 more problems

Ex 1 A light is on the top of a 12 ft pole and a 5'6" tall person is walking away from the pole at a rate of 2 ft/sec

(a) at what rate is his shadow (tip) moving away from the pole when the person is 25 ft from the pole

(b) at what rate is his shadow lengthening when he is 25 ft from the pole

Sol^n (1) Picture & Label



two things are changing

(1) y

(2) $x+y$

(2) Known $\frac{dx}{dt} = 2$

(3) Asked?

(a) $\frac{dx}{dt} + \frac{dy}{dt}$ when $x = 25$

(b) $\frac{dy}{dt}$ when $x = 25$

(4) Relate Variables

Here, we have similar triangles

so $\frac{y}{5\frac{1}{2}} = \frac{x+y}{12} \Rightarrow 12y = \frac{11}{2}(x+y)$

$\Rightarrow 24y = 11x + 11y \Rightarrow 13y = 11x$

so $y = \frac{11}{13}x$

(5) Relate Rates

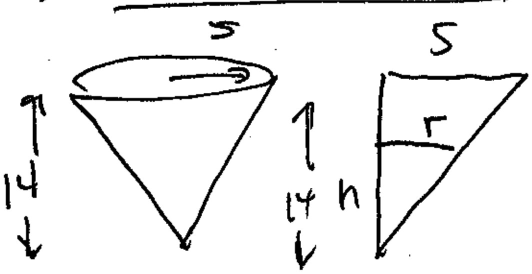
So $\frac{dy}{dt} = \frac{11}{13}\frac{dx}{dt}$ ← note: no x here

so $\frac{dy}{dt} = \frac{11}{13} \cdot 2 = \frac{22}{13}$ ft/sec

$\frac{dx}{dt} + \frac{dy}{dt} = 2 + \frac{22}{13} = \frac{48}{13}$ ft/sec

Ex2 A tank of water in the shape of a cone is leaking at a constant rate of $2\ ft^3/hr$. The base radius is 5 ft & height 14 ft. At what rate is the ~~height~~ depth of water changing when the depth is 6 ft

(1) Picture & Label



$V = \frac{1}{3}\pi r^2 h$ (later)

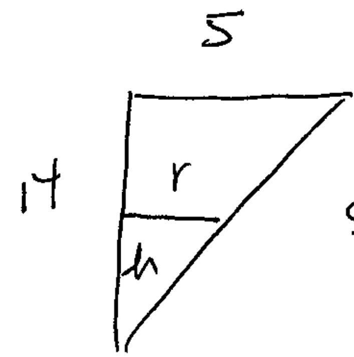
(2) Known $\frac{dV}{dt} = -2$

(3) Asked $\frac{dh}{dt}$ when $h = 6$

(4) Relate variables

Now $V = \frac{1}{3}\pi r^2 h$

Note: we have 2 variables here r & h - we need more info



Similar triangles

so $\frac{r}{5} = \frac{h}{14}$

so $r = \frac{5h}{14}$

so $V = \frac{1}{3}\pi\left(\frac{5h}{14}\right)^2 h = \frac{25\pi}{588}h^3$

Relate Rates

$$\frac{dV}{dt} = \frac{3\cdot 25\pi h^2}{588}\frac{dh}{dt}$$

so $\frac{dh}{dt} = \frac{588}{3\cdot 25\pi h^2}\frac{dV}{dt}$

Sub in what we know

$$\frac{dh}{dt} = \frac{588}{3*25\pi*6^2}(-2) = \frac{-98}{225\pi} \text{ ft/hr.}$$