Math 4315-PDEs

Ex $\quad x u_x - y u_y = u$

if $\quad \cancel{x}\; u_s = u_x x_s + u_y y_s$

Pick $\quad x_s = x, \; y_s = -y$ then $u_s = u$

so $\quad \dfrac{\partial x}{\partial s} = x \qquad \dfrac{\partial x}{x} = \partial s \qquad \ln x = s + \ln a(r)$

$$\Rightarrow \boxed{x = a(r)e^{s}}$$

and $\quad \dfrac{\partial y}{\partial s} = -y$ so $\dfrac{\partial y}{y} = -\partial s \quad \ln y = -s + \ln b(r)$

$$\boxed{y = b(r)e^{-s}}$$

and $\quad \dfrac{\partial u}{\partial s} = u$ so $\dfrac{\partial u}{u} = \partial s \Rightarrow \ln u = s + \ln c(r)$

$$\boxed{u = c(r)e^{s}}$$

Now eliminate s

$$xy = a(r)\,b(r) = A(r) \Rightarrow r = A^{-1}(xy)$$

$$\frac{y}{x} = \frac{c(r)}{a(r)} = B(r) \qquad \frac{y}{x} = B(A^{-1}(xy))$$

Sol^n $\boxed{\frac{y}{x} = f(xy)}$

Now we want to eliminate the s step right away

$$\frac{\partial x}{\partial s} = x, \quad \frac{\partial y}{\partial s} = -y \qquad \frac{\partial u}{\partial s} = u$$

$$\frac{\partial x}{x} = \partial s \qquad \frac{\partial y}{-y} = \partial s \qquad \frac{\partial u}{u} = \partial s$$

so $\frac{\partial x}{x} = -\frac{\partial y}{y} = \frac{\partial u}{u}$ (s is gone!)

when we integrate 1^{st} pair

$$\ln x = -\ln y + \ln A(r)$$

$$xy = A(r)$$

& 1^{st} & 3^{rd}

$$\ln x = \ln u - \ln B(r)$$

$$\Rightarrow \quad \frac{u}{x} = B(r)$$

we essentially treat $A(r) = c_1$, $B(r) = c_2$

& $\quad c_2 = f(c_1)$

$$\Rightarrow \quad \frac{u}{x} = f(xy) \quad \text{as before}$$

so we solve

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{du}{u}$$

Method of characteristics (M.o.C)

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$$

CE $\quad \dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{du}{c}$

Pick 2 pair & integrate

$$\phi_1(x,y,u) = c_1$$

$$\phi_2(x,y,u) = c_2$$

Sol^n $\quad c_2 = f(c_1)$ or $c_1 = f(c_2)$

Ex! $\quad u_x - 3u_y = 2$

CE $\quad \dfrac{dx}{1} = \dfrac{dy}{-3} = \dfrac{du}{2}$

1st p. $\quad +3dx = -dy \Rightarrow 3x + y = c_1$

2nd

$2x = u - c_2 \qquad c_2 = u - 2x$

$c_2 = f(c_1) \Rightarrow \qquad u - 2x = f(3x+y)$

a $\quad u = 2x + f(3x+y) \qquad$ (See Lect 2 Ex 1)

Ex 2

$x u_x - 2y u_y = u$

CE $\quad \dfrac{dx}{x} = \dfrac{dy}{-2y} = \dfrac{du}{u}$

(1) $\quad 2\ln x = -\ln y + \ln c_1 \Rightarrow \quad c_1 = x^2 y$

(2) $\quad \dfrac{dx}{x} = \dfrac{du}{u} \Rightarrow \quad \ln x = +\ln u - \ln c_2$

$c_2 = u/x$

$c_2 = f(c_1) \Rightarrow \qquad \dfrac{u}{x} = f(x^2 y)$

a $\quad u = x f(x^2 y) \quad$ (See Ex 1 Lect 4)

Ex 3 $\quad y u_x + (u-x) u_y = y \qquad u(x,0) = 2x$

ChofC $\quad \dfrac{dx}{y} = \dfrac{dy}{u-x} = \dfrac{du}{y}$

1st pair $\quad dx = du \Rightarrow x = u - c_1 \Rightarrow c_1 = u - x$

2nd $\quad \dfrac{dy}{u-x} = \dfrac{du}{y}$ or $\quad y\,dy = c_1\,du$

$$\frac{y^2}{2} = c_1 u + c_2 \Rightarrow c_2 = y^2/2 - u(u-x)$$

so (A) $\quad u - x = f\left(y^2/2 - u(u-x)\right)$

or (B) $\quad y^2/2 - u(u-x) = g(u-x)$ *

$$0 - 2x(2x-x) = g(2x-x)$$

$$\Rightarrow \quad g(x) = -2x^2$$

Sol $\quad \boxed{y^2/2 - u(u-x) = -2(u-x)^2}$