# Basic Fractional Calculus and Laplace Transforms

Imran Hoque

*Shri Jagdishprasad Jhabarmal Tibrewala University, Vidyanagari, Jhunjhunu,Rajasthan*

***Abstract –*** This research review study paper explores the possibility of applying the Laplace transform for solving fractional calculus from several sources, academic articles and journals. The Laplace transform is very powerful component in engineering, science, and applied mathematics. It permits to transform the fractional calculus into the algebraic equation, so as to solve the algebraic equations to obtain the unknown value as its function, and that further can be processed by applying the Inverse Laplace Transform.

The subject applications of fractional calculus, which means, calculus of integrals as well as derivatives of some arbitrary real and complex order, have possessed seemingly high reputation in the past 30 years, specifically because of their established applications in innumerable diverse fields of engineering and science. Certain areas of contemporary fractional model applications involve Fluid Flow, Dynamical Processes, Diffusive Transport close to Diffusion, Solute Transport in Similar to Porous Structures, Electromagnetic Theory, Viscoelastic Material Theory, Earthquake Dynamics, Dynamical Control Theory Systems, Bioscience, Signal and Optical Processing, Geology, Economics, Astrophysics, Chemical Physics, Statistics, Probability and so on.

## I. INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration of arbitrary order. The subject is as old as the differential calculus and goes back to times when Leibnitz and Newton invented differential calculus.

Fractional calculus has recently been applied in various areas of engineering, science, finance, applied mathematics and bio engineering. However, many researchers remain unaware of this field.They often ask: What is a fractional derivatives? In this talk, several definition of fractional derivatives will be introduced.

The Laplace Transform is a very useful and effective technique for solving ordinary differential equations with constant coefficient, partial differential equations and integral equations under given initial conditions.

The theory of laplace transforms or laplace transformation, also referred to as operational calculus, has an recent years become an essential part of the mathematical background required of engineers, physicists, mathematicians and other scientists. This is because in addition to being of great theoretical interest in itself, laplace transform method provide easy and effective means for the solution of many problems arising in various field of science and engineering.

## II. FRACTIONAL CALCULUS

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number powers or complex number powers of the differentiation operator

$$D = \frac{d}{dx}$$

and the integration operator *J*. (Usually *J* is used instead of *I* to avoid confusion with other *I*-like glyphs and identities.)

$$\sqrt{D} = D^{1/2}$$

In this context, the term *powers* refers to iterative application of a linear operator acting on a function, in some analogy to function composition acting on a variable, e.g., $f^2(x) = f(f(x))$. For example, one may ask the question of meaningfully interpretingas an analog of the functional square root for the differentiation operator, i.e., an expression for some linear operator that when applied *twice* to any function will have the same effect as differentiation.

There are several approaches to the generalization of the notion of differentiation to fractional orders, for example, the Riemann-Liouville, Gr ¨ unwald-Letnikov, Caputo, and generalized functions. The Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real-world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations. Unlike the Riemann-Liouville approach, which derives its definition from repeated integration, the Gr¨ unwald-Letnikov formulation approaches the problem from the derivative side. This approach is mostly used in numerical algorithms. Here, we mention the basic definitions of the Caputo fractional-order integration and differentiation, which are used in the upcoming paper and play the most important role in the theory of differential and integral equation of fractional order. The main advantages of Caputo approach are the initial conditions for fractional differential equations with the Caputo derivatives taking on the same form as for integer order differential equations.

## III. BASIC FRACTIONAL CALCULUS

As far as the motives of classical calculus are concerned, those are derivational and integral of functions which are basically opposite to each other. So when we begin with a function f (t) while putting its derivatives on the left direction as well as on the right direction as with integral, a sequence with two side infinity is obtained.

$$\ldots \frac{d^2 f(t)}{dt^2}, \frac{d f(t)}{dt}, f(t), \int_a^t f(\tau)\,d\tau, \int_a^t \int_a^{\tau_1} f(\tau)\,d\tau\,d\tau_1, \ldots$$

In case of fractional calculus, it attempts of interpolation if the direction. Hence, this pattern conjugates the classical derivatives as well as integrals to make it general and random sequence. A term called differ integral will be discussed, however in a few cases, the term α-derivative (α is a random real number) can also be utilized that refers to an integral when only $\alpha < 0$, otherwise fractional derivative along with fractional integral will be talked.

Several methods are there for defining the differintegral which are named as per their scholars or postulating scientists. For instance, the Grunwald-Letnikov description on differintegral begins with classical descriptions of derivatives and integrals on the basis of infinitesimal division as well as limit. Technological difficulties in association with computational methods as well as evidences and huge level of restricted norms are some of the inconveniences about this technique. But on the brighter side, other methods such as Riemann-Liouville approach cover the outcomes from the earlier approach like a specific instance.

Here in this study, central point is focused on Riemann-Liouville, the Caputo and the Miller-Ross definitions as they're the maximum utilized instances within this implementations. Besides, formulations regarding the circumstances of their equivalences will be done to extract or derive the most significant features. Ultimately, some instances of differintegral will be discussed in reference to fundamental operations.

The fractional calculus can be considered in many ways, a novel topic, once it is only during the last thirty years that it has been the subject of specialized conferences and treatises. Everything has begun with the important applications discovered in numerous diverse and widespread fields in science, engineering and finance. More specifically, we can easily find a direct application of fractional calculus in the study of Fluids Flow, Porous Structures, Control Theory of Dynamical Systems, Rheology Theory, Viscoelasticity, Chemical Physics, Optics, Signal Processing and in many other problems. One can also read on many specialized texts, that the fractional derivatives and integrals are very suitable for modeling the memory properties of various materials and processes that are governed by anomalous diffusion *(term used to describe adiffusion process with a non-linear relationship to time)*.

It is important to understand that the concept of fractional calculus is not a new one, even we already said that it is a novel topic. The history is believed to have emerged from a question raised in the year 1695 by Marquis de L'Hôpital to Gottfried Wilhelm Leibniz. In his letter, L'Hôpital asked about a particular notation Leibniz had used on his publications for the derivative. He used to write.

$$\frac{d^n}{dx^n} f(x)$$

To symbolize the n-th derivative of a function $f$, to $n \in N^* :=$ f{1, 2, . . .}. L'Hôspital posed his question, arguing about the possibility of taking $n = 1{=}2$. In his reply, dated 30 September of 1965, Leibniz wrote back saying

Even Fourier mentions the fractional derivatives but did not give applications or at least examples. So been the first to make applications, N. H. Abel in 1823 [1] studied the fractional calculus in the solution of an integral equation which arises in the formulation of the tautochrone problem: it consist in finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire slides to the lowest point of the wire in the same time regardless of where the bead is placed. The brachistochrone problem deals with the shortest time of slide.

Abel's solution was so elegant that in the guess of many mathematicians, was what attracted the attention of Liouville who made the first major attempt to give a logical definition of the nowadays fractional calculus structure. Liouville's first definition involved.

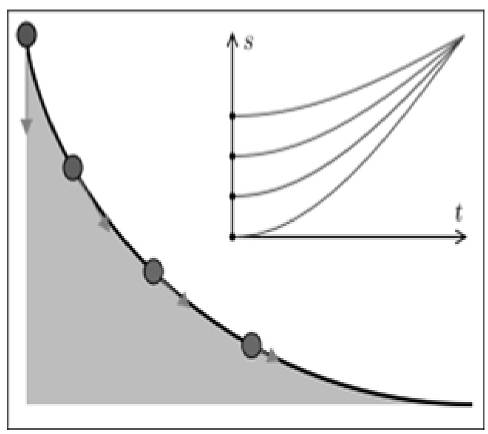



Fig.1: Tautochrone Curve

an infinite series and as we know, the notion of convergence ever would interfere in the definition itself, what was his first obstacle. It was Liouville's second definition that solved the last problem. The notion adopted for the fractional differential of an integrable function $f : [t_0, t_1] \subset R \to R$ was

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d^m}{dt^m}\right)\int_{t_0}^{t}(t-s)^{m-\alpha-1}f(s)\,ds, \qquad t\in[t_0,t_1] $$

Where$\alpha > 0$ and $m$ is the first integer greater or equal than$\alpha$.

As the last advance, in order to present a notion more compatible with the usual theory of differential equations, comes the notion introduced by M. Caputo in 1967 in his celebrated paper [18]. In contrast to the Riemann-Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to deal with the singularity on $t = 0$ and define the fractional order initial conditions, which eventually could be unpleasant to the physics theory. Caputo's definition is illustrated for regular

$$ c_{t_0}D_t^{\alpha} f(t) = {}_{t_0}D_t^{\alpha}\left[f(t) - \sum_{k=0}^{m-1}\left(\frac{d^k f(t)}{dt^k}\right)\Bigg|_{t=t_0}\frac{(t-t_0)^k}{k!}\right], \qquad t\in[t_0,t_1] $$

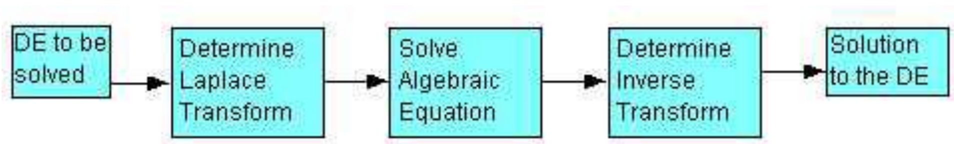
Where we also have that$\alpha > 0$ and $m$ is the first integer greater or equal than.

LAPLACE TRANSFORMS OF DIFFERENTIAL EQUATION

Laplace transform is yet another operational tool for solving constant coefficients linear differential equations. The process of solution consists of three main steps:

1) The given \hard" problem is transformed into a \simple" equation.
2) This simple equation is solved by purely algebraic manipulations.
3) The solution of the simple equation is transformed back to obtain the solution of the given problem.

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration.




The Laplace transform can be used to solve differential equations. Besides being a different and efficient alternative to variation of parameters and undetermined coefficients, the Laplace method is particularly advantageous for input terms that are piecewise-defined, periodic or impulsive.

**DEFINITION:**

Let,$F(t)$ be a function of t defined for all t≥0. Then the laplace transform of $F(t)$, denoted by $L\{f(t)\}$,is defined by,

$$ L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt $$

provided that the integral exists, s is a parameter which may be real or complex.

$L\{F(t)\}$ is clearly a function of s and is briefly written as$f(s)$.

i.e $L\{F(t)\} = f(s)$

Example: Let $f(t) = 1$, then s $F(s) = \frac{1}{s}$, s > 0.

$$ L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st}\,dt = \frac{-1}{s}e^{-st}\Big]_0^\infty $$

The integral is divergent whenever s ≤ 0. However, when s > 0, it converges to

$$ \frac{-1}{s}(0 - e^0) = \frac{-1}{s}(-1) = \frac{1}{s} = F(s). $$

## IV. DISCUSSION & CONCLUSION

A latest generalization functional methods have been generated for the linear differential equation having fractional derivative. This newest generalization is derived from the Caputo fractional derivatives. Hence, it can be stated that the technique applied is highly effective, strong, and efficient for attaining the analytical solution for a big category of equations, linear differential kind of fractional order. Hence, this research review offers a Laplace Transform overview.

The basic application of Laplace Transform is to convert the time functional domain into the frequency domain. The Laplace Transform main properties and its other special functions mentioned include the Laplace Transform, Inverse explanation, which is a very useful and effective mathematical component that simplifies many complex problems in the control and stability zone. In effect, the Laplace Transform is provided incredible applications and help in several power engineering, chemical, electrical, physical and applied sciences.

## V. REFFERENCES


[1]. N. H. Abel, *Solution de Quelques Problèmes A`l'aide D'integráles Définies*, Oeuvres Complètes D'Abel, Christiania, 1881 *(Reprint from the 1823 edition)*.

[2]. R. Adams, *Sobolev Spaces*, Academic press, New York, 1975.

[3]. J. M. Arrieta and A. N. Carvalho, Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations, *Transactionsof the American Mathematical Society* **352**(2000), 285–310.

[4]. M. F. Almeida and L. C. F. Ferreira, Self-similarity, symmetries and asymptotic behavior in Morrey spaces for a fractional wave equation, *Differential IntegralEquations* **25**(9-10) (2012), 957–976.

[5]. H. Amann, On abstract parabolic fundamental solutions, *Journal of the MathematicalSociety of Japan* **39**(1987), 93–116.

[6]. H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, 9–126, in: Schmeisser/Triebel: *Function Spaces, DifferentialOperators and Nonlinear Analysis* **133**, Teubner, Stuttgar, 1993.

[7]. H. Amann, *Linear and Quasilinear Parabolic Problems. Abstract Linear Theory*, Birkhäuser Verlag, 1995.

[8]. J. M. Arrieta and A. N. Carvalho, Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations, *Transactionsof the American Mathematical Society* **352**(2000), 285–310.

[9]. A. Asanov, *Regularization, Uniqueness and Existence of Solutions of Volterra Equationsof the First Kind*, Walter de Gruyter, Utrecht, 1998.

[10]. D. Baleanu and J. I. Trujillo, A new method of finding the fractional Euler–Lagrange and Hamilton equations within Caputo fractional derivatives, *Communications inNonlinear Science and Numerical Simulation* **15**(2010), 1111–1115.