CAP 5993/CAP 4993
Game Theory

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Schedule

- HW4 due 4/13.
- Project presentations on 4/18 and 4/20.
- Project writeup due 4/20.
- Final exam on 4/25.

Project presentations

- Presentations: 10 mins each + 2 mins for questions
- 3 presentations on 4/18
  - I will be wrapping up class material in the first half of class.
- 6 presentations on 4/20
- Send your titles and abstracts when they are ready

Project paper

- Maximum of 10 pages. I will send out formatting information on Thursday.

Open-bid ascending auction (English auction)

- This is the most common public auction. It is characterized by an auctioneer who publicly declares the price of the object offered for sale. The opening price is low, and as long as there are at least two buyers willing to pay the declared price, the auctioneer raises the price (either in discrete jumps, or in a continuous manner using a clock). Each buyer raises a hand as long as he is willing to pay the last price that the auctioneer has declared. The auction ends when all hands except one have been lowered, and the object is sold to the last buyer whose hand is still raised, at the last price declared by the auctioneer. If the auction ends in a draw, a previously agreed rule (such as tossing a coin) is employed to determine who wins the object, which is then sold to the winner at the price that was current when they lowered their hands.
  - Used for web-based auctions and auctions of works of art.
  - https://www.youtube.com/watch?v=UJafks_lAc8

Open-bid descending auction (Dutch auction)

- The auctioneer begins by declaring a very high price, higher than any buyer could be expected to pay. As long as no buyer is willing to pay the last declared price, the auctioneer lowers the declared price (either in discrete jumps or in a continuous manner using a clock), up to the point at which at least one buyer is willing to pay the declared price and indicates his readiness by raising his hand or pressing a button to stop the clock. If the price drops below a previously declared minimum, the auction is stopped, and the object on offer is not sold. Similarly to the English auction, a previously agreed rule is employed to determine who wins the auction if two or more buyers stop the clock at the same time.
  - The flower auction at the Aalsmeer Flower Exchange, near Amsterdam, is conducted using this method.
  - https://www.youtube.com/watch?v=SkehTvMHiiw

Sealed-bid first-price auction

- In this method, every buyer in the auction submits a sealed envelope containing the price he is willing to pay for the offered object. After all buyers have submitted their offers, the auctioneer opens the envelopes and reads the offers they contain. The buyer who has submitted the highest bid wins the offered object, and pays the price that he has bid. A previously agreed rule determines how to resolve draws.

- Theorem: The sealed-bid first price auction is equivalent to the open-bid descending auction.

Sealed-bid second price auction (Vickrey auction)

- The sealed-bid second-price auction method is similar to the first-price sealed-bid auction method, except that the winner of the auction, i.e., buyer who submitted the highest bid, pays the *second*-highest price among the bid prices for the offered object. A previously agreed-upon rule determines the winner in case of a draw, with a winner in this case paying what he bid (which is, in the case of a draw, also the second-highest bid).

- Theorem: The sealed-bid first price auction is equivalent to the open-bid descending auction: both describe the same strategic-form game, with the same strategy sets and payoff functions.
  - Proof: For both, the strategy sets are the set of probability distributions (i.e., *measurable functions* in this setting) over bid sizes for each private valuation v_i in V_i. $\beta_i: V_i \rightarrow [0,\infty)$. A strategy of buyer i is a function detailing how he should play at each of his information sets. An open descending auction ends when the clock is stopped. Hence his only information consists of the current price. A strategy of buyer i then only needs to determine, for each of his possible private values, the announced price at which he will stop the clock (if no other buyer has stopped the clock before that price has been announced).
  - In both auctions, every strategy vector $\beta = (\beta_i)$ leads to the same outcome in both auctions: in a sealed-bid first-price auction the winning buyer is the one who submits the highest bid $\max_i \beta_i(v_i)$, and the price he pays for the auctioned object is his bid. In an open descending auction, the winning buyer is the one who stops the clock at the price $\max_i \beta_i(v_i)$, and the price he pays for the auctioned object is that price. It follows that both types of auctions correspond to the same strategic-form game.

Equivalence between open-bid descending auction and sealed-bid first-price auction

- Note that this equivalence does not depend on any assumptions on the information that each buyer has regarding the other buyers, their preferences, and their identities, or even the number of other buyers. Similarly it does not depend on the assumption that the private values of the buyers are independent.

(Bayesian) equilibrium

- Definition: A strategy vector β^* is a (Bayesian) equilibrium if for every buyer i and every private value v_i,
  - $u_i(\beta^*) >= u_i(b_i, \beta^*_{-i}; v_i)$ for all b_i in $[0,\infty)$

- In other words, β^* is an equilibrium if no buyer i with private value v_i can profit by deviating from his equilibrium bid $\beta^*_i(v_i)$ to another bid b_i.

- Theorem: in a sealed-bid second-price auction, the strategy of buyer i in which he bids his private value weakly dominates all his other strategies.
- So the auction will proceed as follows:
  - Each bidder will bid $b_i = v_i$.
  - The winner will be the buyer whose private valuation of the object is the highest. The price paid by the winning buyer (i.e., the object's *sale price*) is the second-highest private value. If several buyers share the same maximal bid, one of them, selected randomly by a fair lottery, will get the object, and will pay his private value (which in this special case is also the second-highest bid and his profit will therefore be 0).

Sealed-bid second-price auction equilibrium strategies

- Proof (sketch): Consider a buyer i whose private value is v_i. Divide the set of strategies available to her, $S_i = [0,\infty)$, into three subsets:
  - The strategies in which her bid is less than v_i: $[0,v_i)$
  - The strategies in which her bid is equal to v_i: $\{v_i\}$
  - The strategies in which her bid is higher than v_i: (v_i,∞)
- Full proof in the textbook

- Theorem: In a sealed-bid second-price auction, the strategy in which every buyer's bid equals his private value is an equilibrium.
  - Proof: Previous theorem showed that bidding the true value is a dominant strategy. Previous corollary showed that a vector of dominant strategies is an equilibrium, and therefore this strategy vector is a Bayesian equilibrium.

Equilibrium of English auction

- Theorem: In an open ascending (English) auction), the strategy of buyer i that calls on him to lower his hand when the declared price reaches his private value, weakly dominates all his other strategies.
  - Proof: As long as the declared price is lower than buyer i's private value, he receives 0 with certainty if he quits the auction. On the other hand, if he continues to bid, he stands to receive a positive profit (and certainly cannot lose). When the declared price equals buyer i's private value, if he quits he receives 0 with certainty, but if he continues to bid he may win the auction and end up paying more for the object than he values it for. Here we are relying on the fact that buyer i knows his private value, and that this value is independent of the values of the other buyers, so that the information given by the timing that the other buyers choose for quitting the auction is irrelevant to his strategic considerations.

- Theorem: In an open-bid ascending auction, the strategy vector in which every buyer lowers his hand when the declared price equals his private value is an equilibrium.
  - Note that the winner of the object is the buyer with the highest private value and the selling price is the second highest private value. This is the same allocation and the same payment as in the dominant strategy equilibrium of the sealed-bid second-price auction just established.

- Note that there are also other equilibria in sealed-bid second-price auctions, in addition to the equilibrium in which every buyer's bid equals his private value. For example, if the private values of two buyers are independent and uniformly distributed over the interval [0,1], the strategy vector in which buyer 1's bid is $b_1=1$ (for every private value v_1), and buyer 2's bid is $b_2=0$ (for every private value v_2) is an equilibrium.

Summary

- Open-bid ascending auction (English auction)
  - Lowering hand when price reaches private valuation is equilibrium
  - (similar to ebay auction)
- Open-bid descending auction (Dutch auction)
  - Strategically equivalent to sealed-bid first-price
- Sealed-bid first-price auction
  - Strategically equivalent to open-bid descending
- Sealed-bid second-price auction (Vickrey auction)
  - Truthfully bidding private valuation is equilibrium

- Open-bid descending and sealed-bid first-price auctions have no dominant strategies, so can’t make general claim about one strategy always being optimal. For these, we will analyze a common model where we assume that the private valuations are drawn independently according to a distribution (where the distribution is common knowledge). For example, they could all be drawn independently and uniformly form [0,1].

First vs. second price sealed-bid auction

The following table compares FPSBA to **sealed-bid second-price auction** (SPSBA):

Auction:	First-price	Second-price
Winner:	Agent with highest bid	Agent with highest bid
Winner pays:	Winner's bid	Second-highest bid
Loser pays:	0	0
Dominant strategy:	No dominant strategy	Bidding truthfully is dominant strategy[6]
Bayesian Nash equilibrium[7]	Bidder i bids $\frac{n-1}{n}v_i$	Bidder i truthfully bids v_i
Auctioneer's revenue[7]	$\frac{n-1}{n+1}$	$\frac{n-1}{n+1}$

The auctioneer's revenue is calculated in the example case, in which the valuations of the agents are drawn independently and uniformly at random from [0,1]. As an example, when there are $n = 2$ agents:

- In a first-price auction, the auctioneer receives the maximum of the two equilibrium bids, which is $\max(a/2, b/2)$.
- In a second-price auction, the auctioneer receives the minimum of the two truthful bids, which is $\min(a, b)$.

In both cases, the auctioneer's *expected* revenue is 1/3.

- Suppose there are two bidders, Alice and Bob, whose valuations are both drawn uniformly from [0,1]. What are their equilibrium strategies?

each buyer, in his strategy ... to each other and to him.

12.5.1 Analyzing auctions: an example

Definition 12.14 *In a symmetric auction with independent private values, an equilibrium* $(\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$ *is called a* symmetric equilibrium *if* $\beta_i^* = \beta_j^*$ *for all* $1 \le i, j \le n$; *that is, all buyers implement the same strategy.*

When $\beta^* = (\beta_i^*)_{i \in N}$ is a symmetric equilibrium, we abuse notations and denote the common strategy also by β^*, that is, $\beta^* = \beta_i^*$ for every $i \in N$. Such a strategy is called a **symmetric equilibrium strategy**. We will denote by β_{-i}^* the vector of strategies in which all buyers except buyer i implement strategy β^*. We will sometimes denote the symmetric equilibrium strategy also by β_i^* when we want to focus on the strategy implemented by buyer i.

Example 12.15 **Two buyers with uniformly distributed private values**[3] Suppose that there are two buyers, and that V_i has uniform distribution over $[0, 1]$ for $i = 1, 2$ (and by Assumption (A4) V_1 and V_2 are independent). We will show that in a sealed-bid first-price auction the following strategy is a symmetric equilibrium:

$$\beta_i^*(v_i) = \frac{v_i}{2}, \qquad i = 1, 2. \tag{12.11}$$

This equilibrium calls on each buyer to submit a bid that is half of his private value. Suppose that buyer 2 implements this strategy. Then if buyer 1's private value is v_1, and her submitted bid is b_1, her expected profit is

$$u_1(b_1, \beta_2^*; v_1) = u_1\left(b_1, \frac{V_2}{2}; v_1\right) \tag{12.12}$$

$$= \mathbf{P}\left(b_1 > \frac{V_2}{2}\right)(v_1 - b_1) \tag{12.13}$$

$$= \mathbf{P}(2b_1 > V_2)(v_1 - b_1) \tag{12.14}$$

$$= \min\{2b_1, 1\}(v_1 - b_1). \tag{12.15}$$

This function is quadratic over the interval $b_1 \in [0, \frac{1}{2}]$ (attaining its maximum at $b_1 = \frac{v_1}{2}$), and linear, with a negative slope, when $b_1 \ge \frac{1}{2}$. The graph of the function $b_1 \mapsto u_1(b_1, \beta_2^*; v_1)$ is shown in Figure 12.1 for the case $v_1 \le \frac{1}{2}$ and the case $v_1 > \frac{1}{2}$.

3 This example also appears on page 412 in Chapter 10.

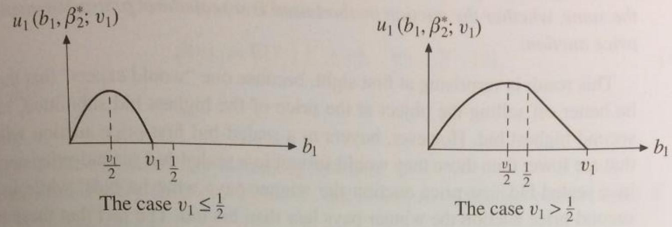

Figure 12.1 The payoff to buyer 1, as a function of b_1, when buyer 2 implements β_2^*

In both cases, the function attains its maximum at the point $b_1 = \frac{v_1}{2}$. This implies that $b_1^*(v_1) = \frac{v_1}{2}$ is the best response to β_2^*, which in turn means that the strategy vector $\beta^* = (\beta_1^*, \beta_2^*)$ is a symmetric equilibrium.

We note that from our results so far we can observe that different auction methods have different equilibria:

- In the sealed-bid first-price auction in Example 12.15, a symmetric equilibrium is given by $\beta_i^*(v_i) = \frac{v_i}{2}$.
- In a sealed-bid second-price auction, a symmetric equilibrium is given by $\beta_i^*(v_i) = v_i$ (Theorem 12.9, and Exercise 12.3).

Which auction method is preferable from the perspective of the seller? To answer this question, we need to calculate the seller's expected revenue in each of the two auction methods. The seller's expected revenue equals the expected sale price. At the equilibrium that we have calculated, the expected sale price is

$$\mathbf{E}\left[\max\left\{\frac{V_1}{2}, \frac{V_2}{2}\right\}\right] = \frac{1}{2}\mathbf{E}[\max\{V_1, V_2\}]. \qquad (12.16)$$

Denote $Z := \max\{V_1, V_2\}$. Since V_1 and V_2 are independent, and have uniform distribution over $[0, 1]$, the cumulative distribution function of Z is

$$F_Z(z) = \mathbf{P}(Z \leq z) = \mathbf{P}(\max\{V_1, V_2\} \leq z) = \mathbf{P}(V_1 \leq z) \times \mathbf{P}(V_2 \leq z) = z^2. \qquad (12.17)$$

It follows that the density function of Z is

$$f_Z(z) = \begin{cases} 2z & \text{if } 0 \leq z \leq 1, \\ 0 & \text{otherwise.} \end{cases} \qquad (12.18)$$

We deduce from this that the expected revenue is

$$\tfrac{1}{2}\mathbf{E}[Z] = \tfrac{1}{2}\int_0^1 z f_Z(z)\mathrm{d}z = \int_0^1 z^2 \mathrm{d}z = \tfrac{1}{3}. \qquad (12.19)$$

The seller's expected revenue in a sealed-bid second-price auction is given by $\mathbf{E}[\min\{V_1, V_2\}]$. Note that

$$\min\{V_1, V_2\} + \max\{V_1, V_2\} = V_1 + V_2, \qquad (12.20)$$

and hence

$$\mathbf{E}[\min\{V_1, V_2\}] + \mathbf{E}[\max\{V_1, V_2\}] = \mathbf{E}[V_1] + \mathbf{E}[V_2] = \tfrac{1}{2} + \tfrac{1}{2} = 1. \qquad (12.21)$$

We have already calculated that $\mathbf{E}[\max\{V_1, V_2\}] = \mathbf{E}[Z] = \frac{2}{3}$, so $\mathbf{E}[\min\{V_1, V_2\}] = \frac{1}{3}$. In other words, the seller's expected revenue in a sealed-bid second-price auction is $\frac{1}{3}$. ◄

- Corollary: In the example, in equilibrium, the expected revenue of the seller is the same, whether the auction method used is a sealed-bid first-price auction or second-price auction.

- This result is surprising at first sight, because one “would expect” that the seller would be better off selling the object at the price of the highest bid submitted, rather than the second-highest bid. However, buyers in a sealed-bid first-price auction will submit bits that are lower than those they would submit in a sealed-bid second-price auction, because in a sealed-bid first-price auction the winner pays what he bids, while in a sealed-bid second-price auction the winner pays less than his bid. The fact that these two opposing elements (on one hand, the sale price in a sealed-bid first-price auction is the highest bid, while on the other hand, bids are lower in a sealed-bid first-price auction) cancel each other out and lead to the same expected revenue, is a mathematical result that is far from self-evident.

- Corollary: In the example, all four auction methods presented, the sealed-bid first-price auction, sealed-bid second price auction, open-bid ascending auction, and open-bid descending auction yield the seller the same expected revenue in equilibrium.

- This result is generalized in the Revenue Equivalence Theorem.

Assignment

- Reading for next class: chapter 22 from main textbook.
- Homework 4 due Thursday.