Calculus 3 - Volumes

In Calculus 1 we considered the area problem. Find the area under the curve $y = f(x)$ on the interval $[a, b]$. To do this, we broke the interval up into smaller segments, approximated the area on each segment with a rectangle, added the rectangles up, and then took the limit as the number of rectangle went to infinity and the thickness of each rectangle went to zero.

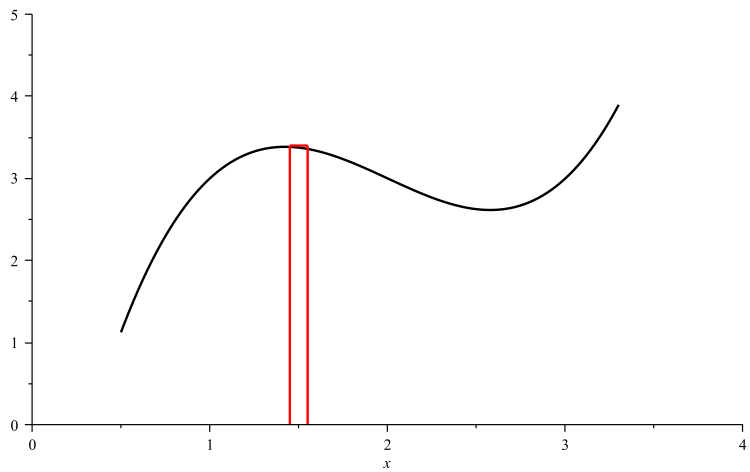


Figure 1: $y = f(x)$

Mathematically: We subdivide the interval

$$a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_n = b. \tag{1}$$

Let

$$\Delta x_i = x_{i-1} - x_i. \tag{2}$$

Pick x_i^* so that

$$x_i^* \in [x_{i-1}, x_i]. \tag{3}$$

Height of the i^{th} rectangle

$$h_i = f(x_i^*). \tag{4}$$

Area of this rectangle

$$A_i = f(x_i^*)\Delta x_i. \tag{5}$$

Add up the rectangles

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i^*)\Delta_i. \tag{6}$$

Then take the limit so

$$A = \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{7}$$

and we gave this Riemann sum a name - a definite integral

$$A = \int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i^*)\Delta x_i. \tag{8}$$

So now we consider the volume problem. Find the volume under the surface $z = f(x, y)$ on the interval $[a, b] \times [c, d]$. The process is the same thing as in the area problem. We approximate the volume with small rectangular boxes.

Mathematically: Subdivide the interval

$$\begin{aligned} a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_m = b \\ c = y_0 < y_1 < y_2 < \cdots < y_{j-1} < y_j < \cdots < y_n = d. \end{aligned} \tag{9}$$

Let

$$\Delta x_i = x_{i-1} - x_i, \quad \Delta y_j = y_{j-1} - y_j, \tag{10}$$

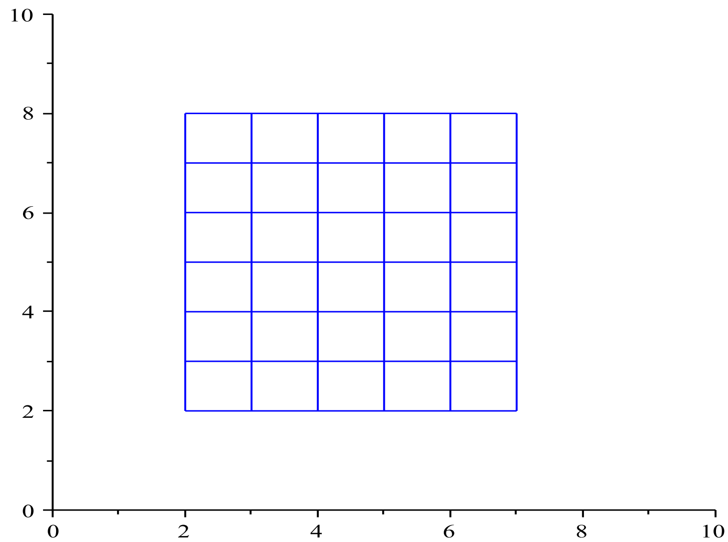


Figure 2: Grid

Pick (x_i^*, y_j^*) so that

$$(x_i^*, y_j^*) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]. \tag{11}$$

Height of the the rectangle box

$$h_{ij} = f(x_i^*, y_j^*). \tag{12}$$

The volume of this rectangle box is

$$V_{ij} = f(x_i^*, y_j^*)\Delta x_i \Delta y_j. \tag{13}$$

Add up the boxes

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j. \tag{14}$$

Then take the limit so

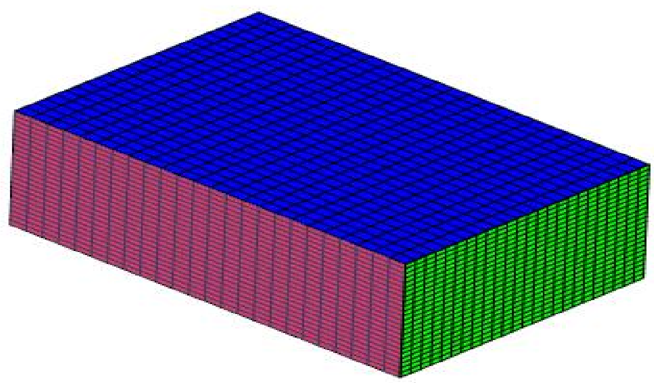
$$\lim_{\substack{m,n\to\infty \\ \Delta x_i, \Delta y_j \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j \tag{15}$$

and we give this Riemann sum a name - a double integral

$$\int_c^d \int_a^b f(x,y)\,dxdy = \lim_{\substack{m,n\to\infty \\ \Delta x_i, \Delta y_j \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j \tag{16}$$

Example 1.

Find the volume under $z = 1$ for $[0,2] \times [0,3]$. Well, we see right away that the volume is 6. Let us use the double integral. So here



$$V = \int_0^3 \int_0^2 1 dxdy. \tag{17}$$

As with partial derivatives, when integrating one variable, we hold the other constant, we do the same with double integral.

$$\begin{aligned} V &= \int_0^3 \left(\int_0^2 1 dx \right) dy \\ &= \int_0^3 \left(x \Big|_0^2 \right) dy \\ &= \int_0^3 2 \, dy \\ &= 2y \Big|_0^3 \\ &= 6. \end{aligned} \tag{18}$$

Switching the Order of Limits

We could also have done

$$\begin{aligned} V &= \int_0^2 \left(\int_0^3 1 dy \right) dx \\ &= \int_0^2 \left(y \Big|_0^3 \right) dx \\ &= \int_0^2 3 \, dx \\ &= 3x \Big|_0^2 \\ &= 6. \end{aligned} \tag{19}$$

Integrating when f is not constant

Consider

$$\begin{aligned} V &= \int_0^3 \int_{-1}^1 12x^2y \, dxdy \\ &= \int_0^3 4x^3y \Big|_{x=-1}^{x=1} dy \\ &= \int_0^3 8y \, dy \\ &= 4y^2 \Big|_{y=0}^{y=3} \\ &= 36. \end{aligned} \tag{20}$$

Switching limits

$$\begin{aligned} V &= \int_{-1}^1 \int_0^3 12x^2y \, dydx \\ &= \int_{-1}^1 6x^2y^2 \Big|_0^3 dy \\ &= \int_{-1}^1 54x^2 \, dy \\ &= 18x^3 \Big|_{-1}^1 \\ &= 36. \end{aligned} \tag{21}$$

In fact, if f is continuous on $[a,b] \times [c,d]$ then

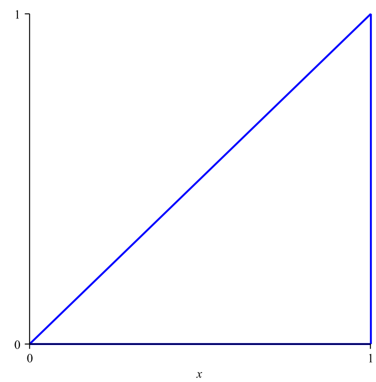
$$\int_a^b \int_c^d f(x,y) \, dydx = \int_c^d \int_a^b f(x,y) \, dxdy \tag{22}$$

Integration over non constant regions

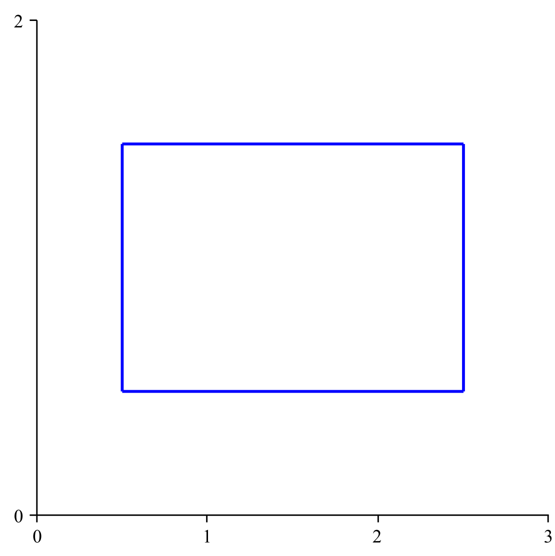
Suppose we wish to set up the double integral

$$\iint_R f(x,y)\,dydx \tag{23}$$

where R is the region below (the lines $y = 0$, $y = x$ and $x = 1$)



To get an idea on how to do this let us first consider the problem when the region is a rectangular box.

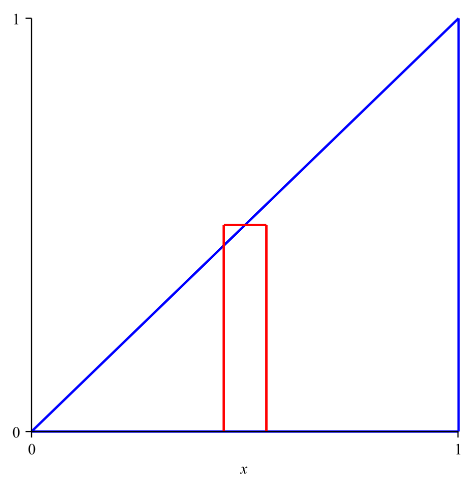


So we have

$$\int_a^b \int_c^d f(x,y)\,dydx = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx \tag{24}$$

In the round bracket x is fixed and y moves from $y = c$ to $y = d$. Now in the triangular region when x is fixed, then y moves from $y = 0$ to $y = x$ and so the limits of integration are

$$\int_a^b \left(\int_0^x f(x,y)\,dy\right) dx \tag{25}$$



Now as the rectangle moves, it moves from $x = 0$ to $x = 1$ and these are the outside limits and so

$$\int_0^1 \left(\int_0^x f(x,y)\,dy\right) dx \tag{26}$$

or simply

$$\int_0^1 \int_0^x f(x,y)\,dydx \tag{27}$$

Example 2. Evaluate

$$\int_0^1 \int_0^x 6y^2 - x^3 y \, dy dx \tag{28}$$

Soln. We first integrate wrt y holding x fixed. So

$$\int_0^1 2y^3 - \frac{x^3 y^2}{2}\Big|_{y=0}^{y=x} dx \tag{29}$$

Then substitute in the limits

$$\int_0^1 \left(2x^3 - \frac{x^3 x^2}{2}\right) - \left(20^3 - \frac{x^3 0^2}{2}\right) dx \tag{30}$$

Then integrate one more time

$$\frac{x^4}{2} - \frac{x^6}{12}\Big|_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}. \tag{31}$$

In general

In general we have

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y) dy dx \tag{32}$$