Application of Particle Filtering for Sequential Inferences of Stochastic Volatility Modeled with Leverage

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Abstract

In this article, we implement the Particle Filtering (PF) algorithm for sequential inferences of the stochastic volatility (SV) modeled with leverage. Our implementation and applications are based on the recent work by Djuric et al. (2012) and the earlier work by Khan (2011) and Liu and West (2001). The standard SV models is a dynamic state space models where the underlying volatility process is assumed an autoregressive hidden process, and the innovations of the state and observed processes are assumed independent and Gaussian white noises with zero mean and unit variance. But, the paper by Djuric et al. (2012) modeled the noise processes as correlated, and called it modeled with leverage. The paper also applied Rao-Blackwellization (RB) method, a variance reduction technique which integrate out the static and nuissance parameters, which reduces the dimension of unknowns and computational complexities. The RB method shows advantage on convergence with the smaller particle size, and improve estimates by reducing the variance of unknowns. The objective is to get real-time inferences of the underlying volatility (log-volatility) as the observations arise sequentially in time.

Keywords: Particle filtering, stochastic volatility, Bayesian, dynamic state space, leverage.

1. Introduction

The increasing number of interlinkages and interdependencies in the global financial system have led to an augmented potential for cascading failure and financial crisis contagion, as evidenced in the failure of LTCM in 1998 and the subprime mortgage crisis in 2008. These events have demonstrated the compelling need for policymakers, regulators and financial industry participants to develop new methods and state-of-the-art tools to better understand, monitor and mitigate global systemic risk. Managing Financial Risk has been an important aspects of the corporations, financial institutions and global central banks, where volatility is one of the key metric to manage and model risk. One of the main focus of our research is

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to model and measure/predict the risk poses on the global financial systems due to increasing electronic trading especially HFT (high frequency trading). The exact sources of the volatility is not known, but, the various factors such as economical, financial, geo-political, natural events, trading volume as well as irrational behaviors of market participants are assumed to create uncertainty in the markets. We apply PF algorithm on SV model since SV is a dynamic predictive model, and can be used to predict/asses the volatility/risk dynamically. The volatility is a measure of *Market Risk*, therefore, modeling and estimating volatility are in great demand among practitioners and academics.

2. Modeling Volatility

In statistics, volatility is defined as the standard deviation (σ_t) of asset price. Study suggest that the variance of time series σ_t^2, is *heteroscedastic* i.e., it changes over time. Hence, it is modeled as a non-stationary process.

$$
\begin{aligned}
r_t &= \mu + \sigma_t v_t && (1)\\
y_t &= \sigma_t v_t && (2)\\
y_t &\sim \mathcal{N}(0, \sigma_t^2) && (3)\\
\sigma_t^2 &= \alpha_0 + \alpha_1 y_{t-1}^2, \quad ARCH(1) && (4)\\
\sigma_t^2 &= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \quad GARCH(1,1) && (5)
\end{aligned}
$$

where $r_t = (P_t/P_{t-1} - 1)$ is standard return series, P_t is price, μ is expected return, $y_t = r_t - \mu$ is mean subtracted return series, and v_t is standard Gaussian noise process, i.e. $v_t \sim \mathcal{N}(0,1)$. The two most widely used volatility models, ARCH (auto regressive conditional heteroscedasticity) and GARCH(generalized auto regressive conditional heteroscedasticity) are developed by Engle (1982) and Bollerslev (1986) respectively to model the volatility.

3. Standard SV Models-Uncorrelated Noise Processes

An alternative hypothesis is that the volatility is a hidden/underlying process, which is not observed directly from the return series. Hence, the formation of the SV models developed by Rosenberg (1972), Clark (1973), Black and Scholes (1972), Taylor (1982), Taylor (2005), Ghysels et al. (1996) and Shephard (1996) as a discrete time dynamic state-space (DSS) models, a class of HMM is considered. The standard SV model is defined as,

$$
\begin{aligned}
x_t &= \beta_1 + \beta_2 x_{t-1} + \sigma_u u_t \quad (state \quad equation) && (6)\\
y_t &= e^{x_t/2} v_t \quad (observation \quad equation) && (7)
\end{aligned}
$$

where $x_t = log(\sigma_t^2), x_t \in \mathbb{R}$ is log-volatility, u_t and v_t are independent and identically distributed standard Gaussian noise process, i.e., $u_t, v_t \sim \mathcal{N}(0,1)$, and β_1, β_2, and σ_u are unknown static but nuissance parameters. Objective is to get sequential inferences of x_t as y_t is observed dynamically in time.

4. Leverage SV Models-Correlated Noise Processes

It has been observed that there is a negative correlation between volatility and return series, and modeling this phenomena is called Leverage SV models (Black, 1976; Nelson, 1991; Jacquier et al., 2004; yu, 2005; Djuric et al., 2012). When the prices decrease dramatically and the returns are negative, the volatility spikes, as seen in 2008-2009 during the market crash when the VIX reached all time high. Hence, the SV models with correlated noise processes/shocks of state and observations are defined as,

$$u_t = \rho v_{t-1} + \sqrt{1-\rho^2}u'_t \tag{8}$$

$$\begin{aligned} x_t &= \beta_1 + \beta_2 x_{t-1} + \sigma_u \rho v_{t-1} + \sigma_u \sqrt{1-\rho^2}u'_t \\ &= \beta_1 + \beta_2 x_{t-1} + \beta_3 y_{t-1} e^{-\frac{x_{t-1}}{2}} + \zeta u'_t \quad (state) \end{aligned} \tag{9}$$

$$y_t = e^{x_t/2} v_t \quad (observation) \tag{10}$$

where $corr(u_t, v_{t-1}) = \rho$, $\beta_3 = \sigma_u \rho$, $\zeta = \sigma_u \sqrt{1-\rho^2}$ and u'_t is another standard Gaussian noise process, independent of v_{t-1} and u_t. The unknown parameter vector $\boldsymbol{\theta} = [\beta_1\ \beta_2\ \sigma_u\ \rho]$. We observe that in this leverage models, x_t has autocorrelation with x_{t-1} and y_{t-1}, which has more intuitive explanation.

5. Rao-Blackwellization(RB)

The RB method transforms an ordinary estimator into an improved estimator based on Rao-Blackwell theorem (Lehmann, 1991). It reduces the variance of the estimate based on mean-squared error (MSE) criterion. The RB (Casella and Robert, 1996; Doucet et al., 2000) also refers to dimensionality reduction in unknown space, such as state space in DSS models. Due to the presence of unknown static parameters $\theta = [\beta_1\ \beta_2\ \sigma_u\ \rho]^\top$, the full joint posterior PDF has the form as $p(x_t, \theta | x_{0:t-1}, y_{1:t-1})$. Hence, it requires MC sampling from joint posterior, i.e, from state and parameter spaces respectively. Since the parameters are nuisance, if one can integrate out the parameter vector θ, the resultant distribution is the desired filtering PDF $p(x_t|x_{0:t-1}, y_{1:t-1})$, which marginalizes the joint posterior PDF by reducing total sampling space. Consequently, this reduction in dimension reduces the variance of the estimate and improves accuracy. This process is also called RB.

$$\begin{aligned} p(x_t|x_{0:t-1}, y_{1:t-1}) &= \int_\theta p(x_t, \theta | x_{0:t-1}, y_{1:t-1}) \mathrm{d}\theta \\ &= \int_\theta p(x_t|\theta, x_{0:t-1}, y_{1:t-1}) p(\theta | x_{0:t-1}, y_{1:t-1}) \mathrm{d}\theta \\ &\propto t_\nu(m, r) \end{aligned} \tag{11}$$

where $t_\nu(m, r)$ is non-standard t-PDF with ν degrees of freedom (df), m=mean and r=variance. We marginalize parameter space by using implied integration method, and hence the marginal filtering PDF becomes non-standard t-PDF as described above.

6. Particle Filtering- A Sequential Bayesian Filtering

Particle Filtering (PF) or Sequential Monte Carlo (SMC) is a simulation based methods based on Bayes' theorem (Gordon et al., 1993; Doucet et al., 2001, 2000; Arulampalam et al., 2002; Maskell, 2004; Djuric et al., 2003; Djuric and Bugallo, 2009). When the DSS systems are linear and Gaussian, the Kalman filter (Kalman, 1960) is the optimal solution. But, most real world dynamical systems are non-linear and non-Gaussian, and obtaining sequential inferences of their hidden states has been a challenge. Various extensions of Kalman filter such as extended Kalman filter (Anderson and Moore, 1979) and Gaussian sum filter (Sorenson and Alspach, 1971) are also being used for non-linear systems. But, when non-linearity is too high these methods perform poorly. For non-linear and/or non-Gaussian systems, the PF has superior performance over existing methods according to research and literatures. PF has wide range of applications in science and engineering, including statistical signal processing, target tracking, missile guidance, terrain navigation, neural networks, financial modeling, and time series analysis and forecasting. Many real world problem such as in finance, where observations arrive sequentially in time and objective is to get realtime inferences of the unknown state. The PF under Bayesian methodology provides sequential inferences of posterior PDF as observation arrives in time.

6.1. Bayesian Inferences

Let $\boldsymbol{x}_{0:t} \equiv (\boldsymbol{x}_0, ..., \boldsymbol{x}_t)$ and $\boldsymbol{y}_{1:t} \equiv (\boldsymbol{y}_1, ..., \boldsymbol{y}_t)$ are defined as state and observation sequences respectively. We assume that we have some prior knowledge about the unknowns, and all information of the unknowns are available in the posterior PDF. Hence, our objective is to sequentially estimate/predict the posterior PDF (Box and Tiao, 1992; Doucet et al., 2001). The transition from the prior to the posterior PDF defined as,

$$\underbrace{p(\boldsymbol{x}_{0:t}|\boldsymbol{y}_{1:t}, \Psi)}_{posterior} = \frac{\overbrace{p(\boldsymbol{y}_{1:t}|\boldsymbol{x}_{0:t}, \Psi)}^{likelihood}\overbrace{p(\boldsymbol{x}_{0:t}|\Psi)}^{prior}}{\underbrace{p(\boldsymbol{y}_{1:t}|\Psi)}_{evidence}} \tag{12}$$

where Ψ is the assumed model.

The posterior PDF $p(\boldsymbol{x}_{0:t}|\boldsymbol{y}_{1:t})$ and its various features are the main interest in Bayesian Inference. In many applications, when the interest is on sequential estimation of posterior PDF, and often a particular interest on its marginal the so called *filtering* PDF, $p(\boldsymbol{x}_t|\boldsymbol{y}_{1:t})$. The joint posterior PDF can be expressed recursively as,

$$p(x_{0:t}|y_{1:t}) = p(x_{0:t-1}|y_{1:t-1})\frac{p(y_t|x_t)p(x_t|x_{0:t-1}, y_{1:t-1})}{p(y_t|y_{1:t-1})} \tag{13}$$

One of the central idea in PF is to represent the posterior PDF by a *random measure*, a set of weighted samples, also known as *particles*. These weighted particles approximate the posterior PDF $p(\boldsymbol{x}_{0:t}|\boldsymbol{y}_{1:t})$, by the random measure, $\chi_{0:t} = \{\boldsymbol{x}^i_{0:t}, w^i_t\}^N_{i=1}$, where $\{\boldsymbol{x}^i_{0:t}\}^N_{i=1}$ are the set of support points with associated weights $\{w^i_t\}^N_{i=1}$, and $\{\boldsymbol{x}^i_{0:t}\}^N_{i=1}$ are the possible trajectories/realizations of the state up to time instant t. As the number of particles tends to infinity, the random measure, under given some conditions, tends almost surely to the true posterior PDF (Crisan and Doucet, 2002). Mathematically, we express

$$p(\boldsymbol{x}_{0:t}|\boldsymbol{y}_{1:t}) \quad \approx \quad \sum_{i=1}^{N} w^i_t \delta(\boldsymbol{x}_{0:t} - \boldsymbol{x}^i_{0:t}) \tag{14}$$

$$\{\boldsymbol{x}^i_{0:t}, w^i_t\}^N_{i=1} \quad \xrightarrow{a.s} \quad p(\boldsymbol{x}_{0:t}|\boldsymbol{y}_{1:t}) \quad as \quad N \to \infty \tag{15}$$

where $\delta(\cdot)$ is the *Dirac delta* function. These weighted particles constitute the discrete approximation of the true posterior PDF. The weights are computed by using the principle of *importance sampling (IS)* (Geweke, 1989). The particles/samples are generated from the *importance function* $q(\boldsymbol{x}_{0:t}|\boldsymbol{y}_{1:t})$ such that $supp(q) \supset supp(p)$, where $p(\cdot)$ is the generic target posterior PDF. If the importance function can be factorized as,

$$q(\boldsymbol{x}_{0:t}|\boldsymbol{y}_{1:t}) = \underbrace{q(\boldsymbol{x}_{0:t-1}|\boldsymbol{y}_{1:t-1})}_{keep\ existing\ path} \underbrace{q(\boldsymbol{x}_t|\boldsymbol{x}_{0:t-1}, \boldsymbol{y}_{1:t})}_{extend\ path} \tag{16}$$

then it is possible to obtain samples (trajectories) $\boldsymbol{x}^i_{0:t} \sim q(\boldsymbol{x}_{0:t}|\boldsymbol{y}_{1:t})$ up to time t, by augmenting the existing trajectories $\boldsymbol{x}^i_{0:t-1} \sim q(\boldsymbol{x}_{0:t-1}|\boldsymbol{y}_{1:t-1})$ with the current state samples $\boldsymbol{x}^i_t \sim q(\boldsymbol{x}_t|\boldsymbol{x}_{0:t-1}, \boldsymbol{y}_{1:t})$ at time t. Thus, the recursive importance function allows us to express the weight update equation as,

$$\begin{aligned} w^i_t &\propto \frac{p(\boldsymbol{x}^i_{0:t}|\boldsymbol{y}_{1:t})}{q(\boldsymbol{x}^i_{0:t}|\boldsymbol{y}_{1:t})} \\ &\propto w^i_{t-1} \frac{p(\boldsymbol{y}_t|\boldsymbol{x}^i_t) p(\boldsymbol{x}^i_t|\boldsymbol{x}^i_{0:t-1}, \boldsymbol{y}_{1:t-1})}{q(\boldsymbol{x}^i_t|\boldsymbol{x}^i_{0:t-1}, \boldsymbol{y}_{1:t})} \end{aligned} \tag{17}$$

If we sample from the marginal filtering PDF $p(\boldsymbol{x}^i_t|\boldsymbol{x}^i_{0:t-1}, \boldsymbol{y}_{1:t-1})$, then $q(\boldsymbol{x}^i_t|\boldsymbol{x}^i_{0:t-1}, \boldsymbol{y}_{1:t}) = p(\boldsymbol{x}^i_t|\boldsymbol{x}^i_{0:t-1}, \boldsymbol{y}_{1:t-1})$, and consequently our updated weight equation becomes,

$$w^i_t \propto w^i_{t-1} p(\boldsymbol{y}_t|\boldsymbol{x}^i_t) \tag{18}$$

6.2. A Generic PF Algorithm

This algorithm is based on choosing prior distribution as importance function, i.e., $q(x_t|x_{t-1}, y_t) = p(x_t|x_{t-1})$ (Doucet et al., 2001), and with the use of *Sequential Importance Sampling* (SIS) method. We refer it as Standard PF (SPF) algorithm and is defined as:

1. Initialization, $t = 0$
   - for $i = 1, \ldots, N$, sample $x^i_0 \sim p(x_0)$ and set $t = 1$.
2. Importance sampling step
   - for $i = 1, \ldots, N$, sample $x^i_t \sim p(x_t|x^i_{t-1})$ and set $x^i_{0:t} = (\widetilde{x}^i_{0:t-1}, x^i_t)$.
   - for $i = 1, \ldots, N$, evaluate the importance weights, $w^i_t = w^i_{t-1} p(y_t|x^i_t)$.
   - normalize the importance weights: $\widetilde{w}^i_t = \frac{w^i_t}{\sum_{j=1}^N w^j_t}, \quad i = 1, \ldots, N$.
3. Selection step
   - resample with replacement N particles$\{\widetilde{x}^i_{0:t}\}_{i=1}^N$ from the set $\{x^i_{0:t}\}_{i=1}^N$ with the importance weights according to some resampling algorithm.
   - If resampling takes place, set $\widetilde{w}_t = 1/N$.
   - set $t \leftarrow t + 1$ and go to step 2.

There are various PF algorithms of which three are most popular. They are the sampling importance resampling (SIR) filter (which is the SPF algorithm described above), the auxiliary particle filter (APF) and the regularized particle filter (RPF).

7. Simulation Study

7.1. Simulated Observation Series and Underlying Log-volatility with Prediction

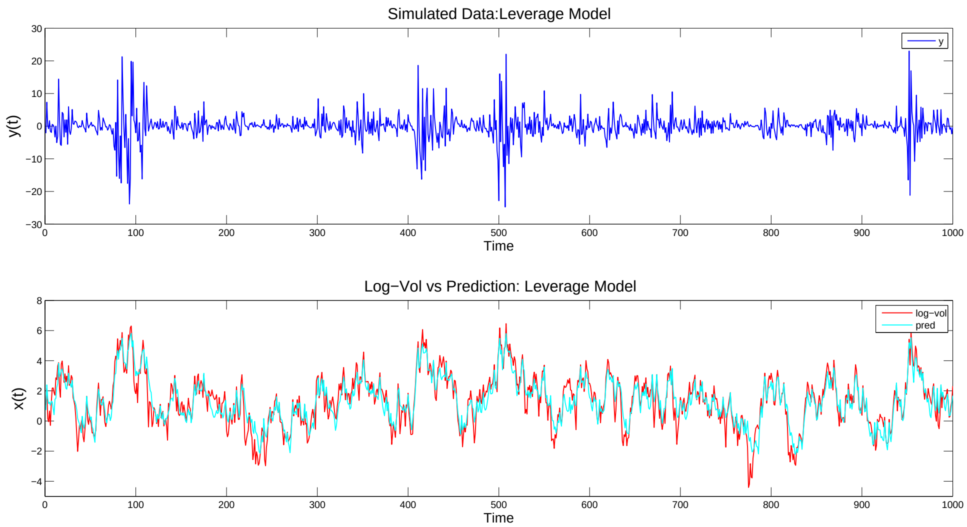

Figure 1: Simulated time series y_t, and underlying log-volatility x_t with prediction (estimate) with the parameters used; $\beta_1 = 0.10, \beta_2 = 0.90, \sigma_u = 0.25$ and $\rho = -0.8$.

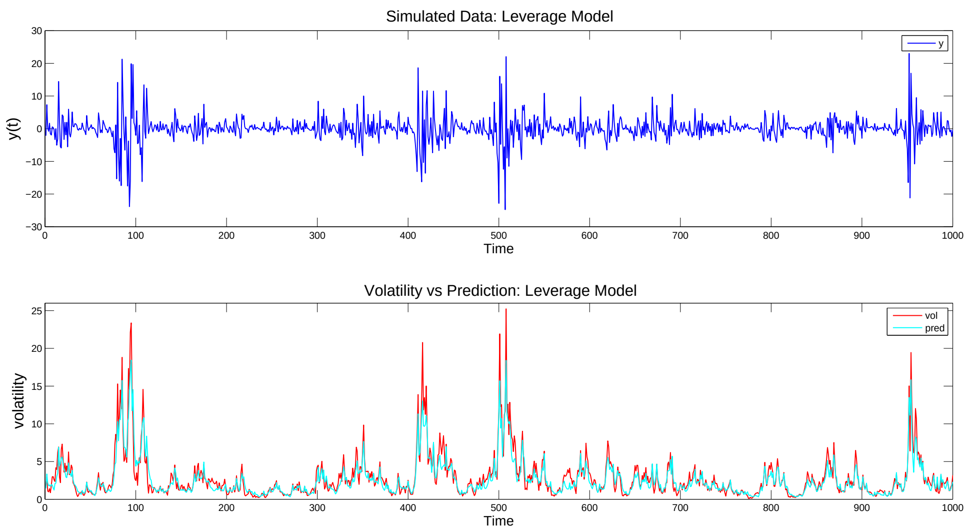

Figure 2: Simulated time series y_t, and underlying volatility σ_t with prediction (estimate) with the parameters used; $\beta_1 = 0.10, \beta_2 = 0.90, \sigma_u = 0.25$ and $\rho = -0.8$.

8. Applications on Real Data

8.1. Log-volatility and Volatility Prediction for S&P500 Return Series:2005-2014

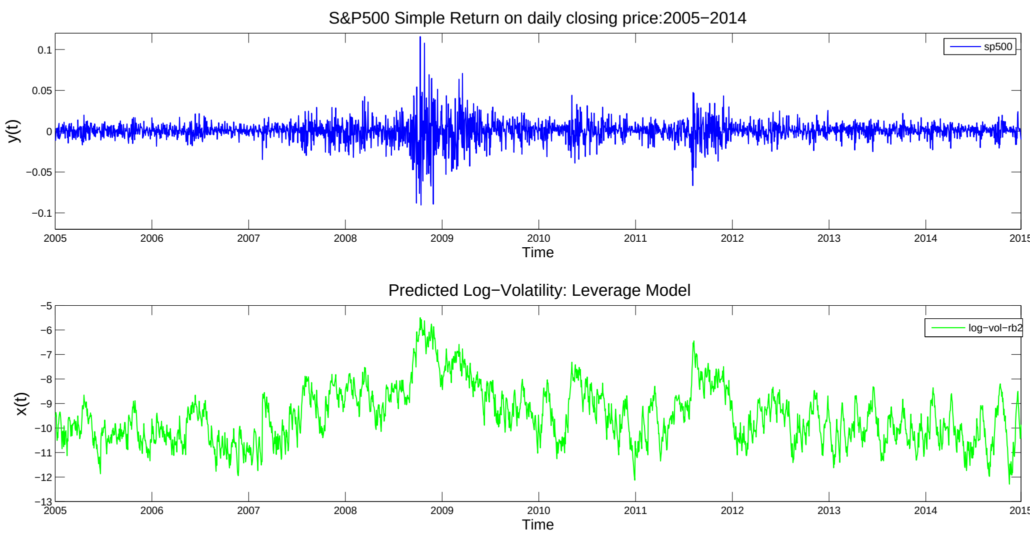

Figure 3: S&P500 Return series y_t, with Predicted underlying log-volatility x_t

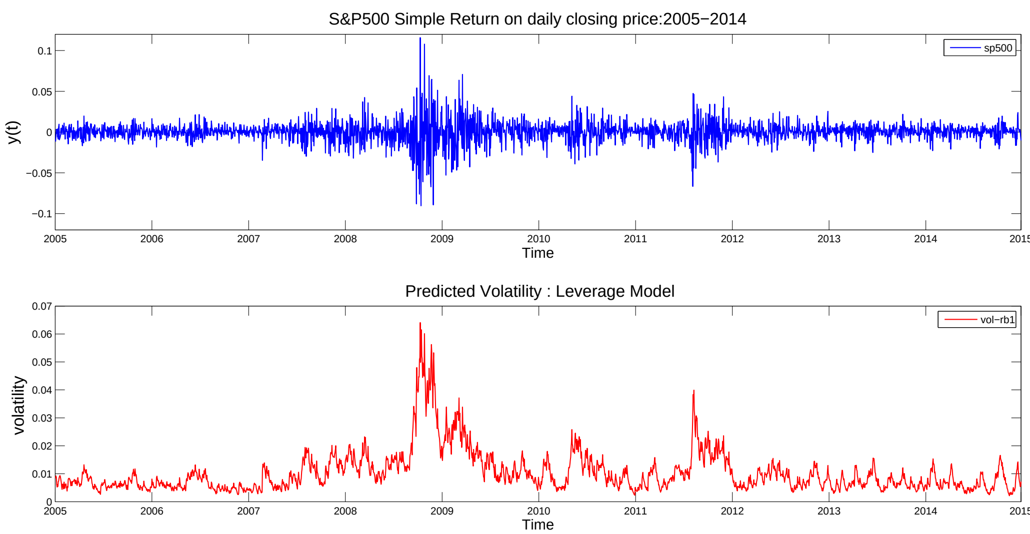

Figure 4: S&P500 Return series y_t, with Predicted underlying Volatility σ_t

8.2. Log-volatility and Volatility Prediction for NASDAQ100 Return Series:2005-2014

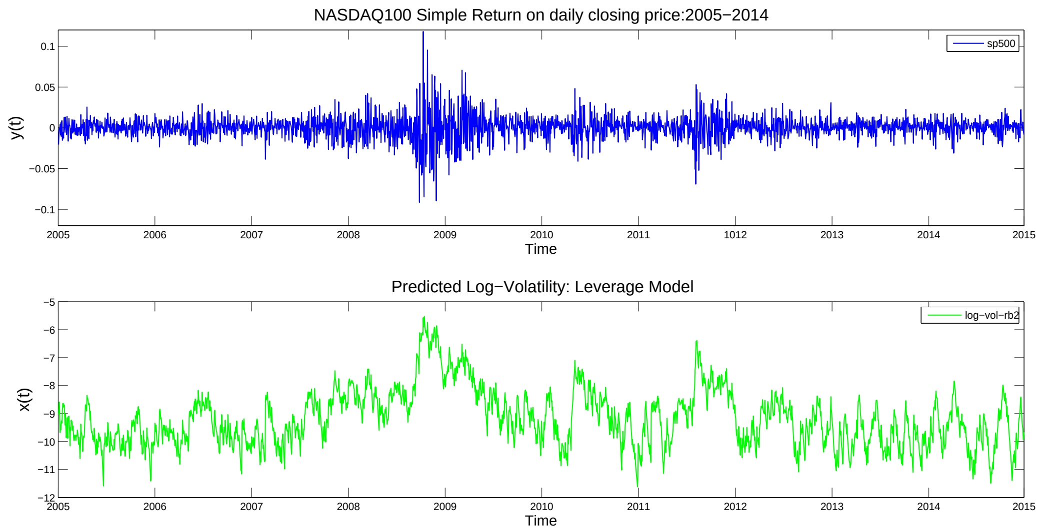

Figure 5: NASDAQ100 Return series y_t, with Predicted underlying log-volatility x_t

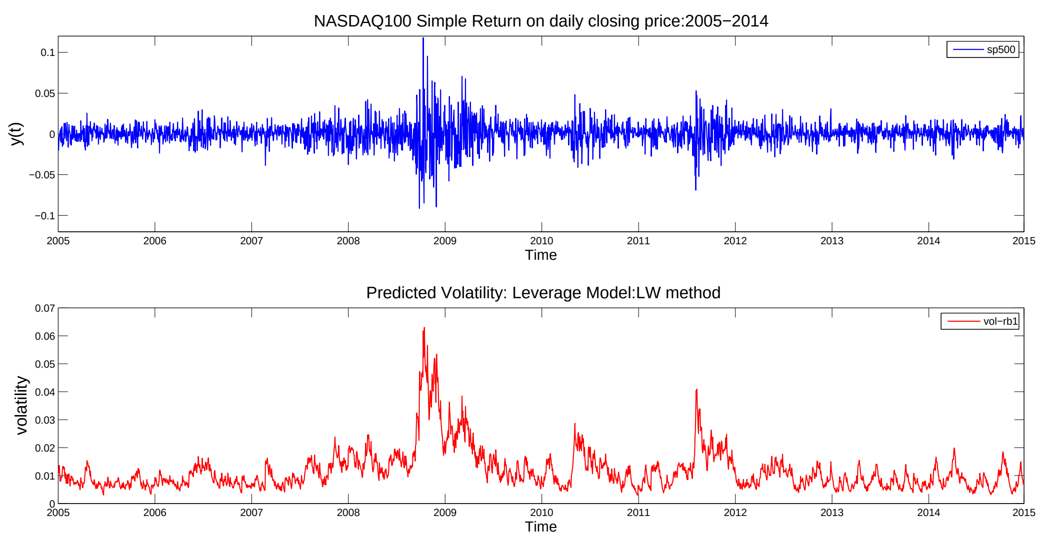

Figure 6: NASDAQ100 Return series y_t, with Predicted underlying Volatility σ_t

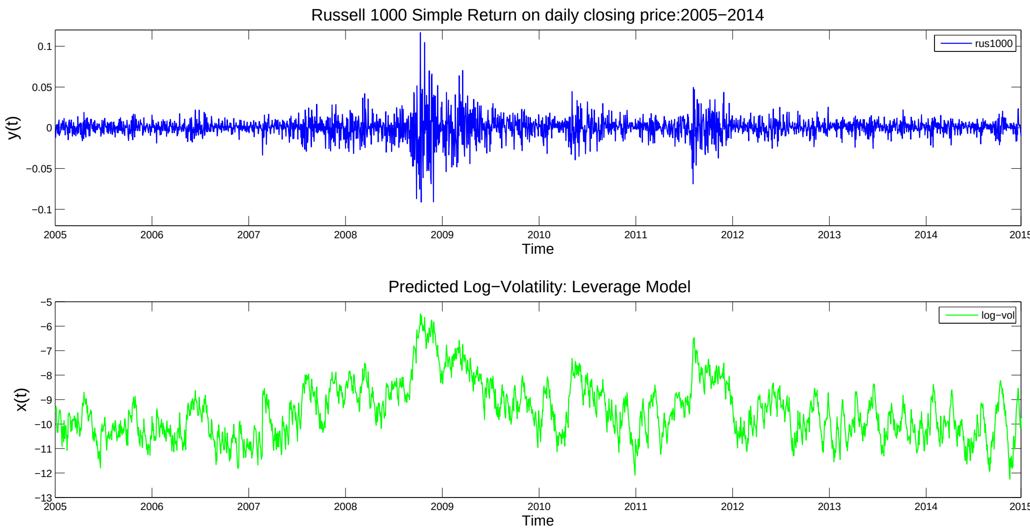

Figure 7: Russell 1000 Return series y_t, with Predicted underlying log-volatility x_t

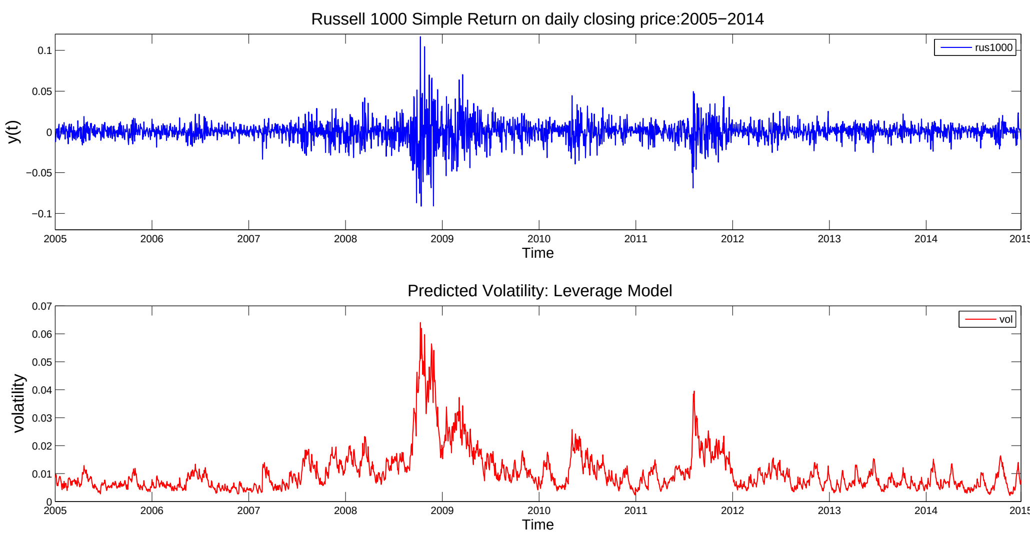

Figure 8: Russell 1000 Return series y_t, with Predicted underlying Volatility σ_t

8.4. *Log-volatility and Volatility Prediction for Russell 2000 Return Series:2005-2014*

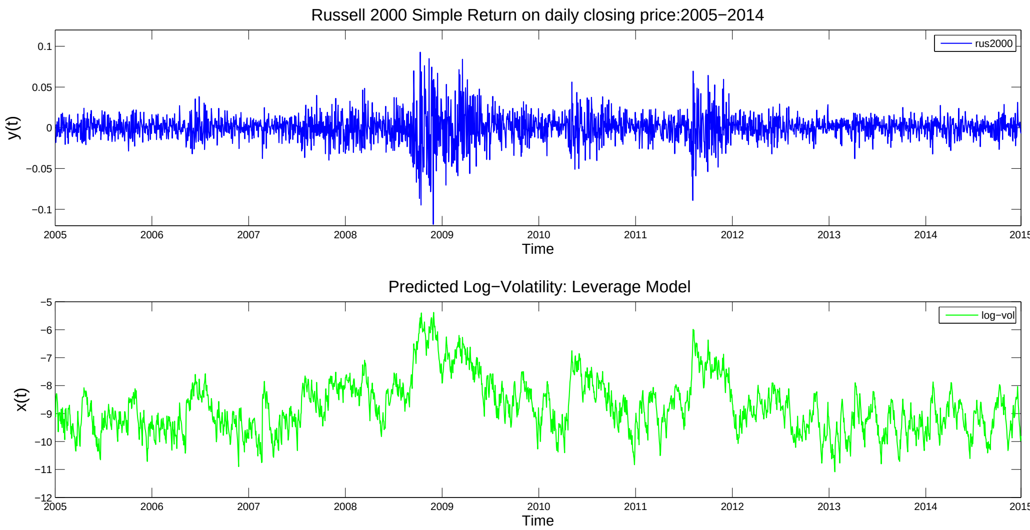

Figure 9: Russell 2000 Return series y_t, with Predicted underlying log-volatility x_t

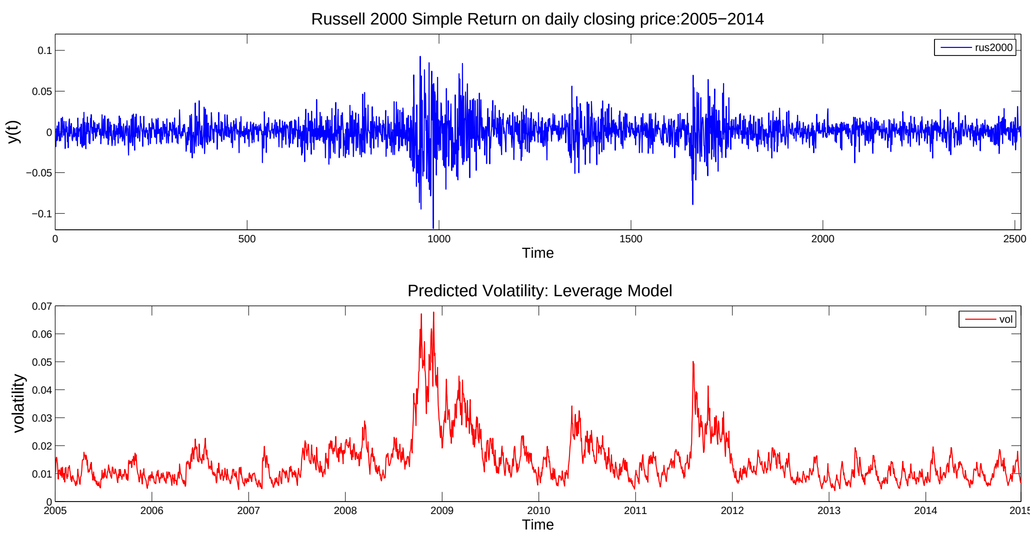

Figure 10: Russell 2000 Return series y_t, with Predicted underlying Volatility σ_t

9. Histograms of S&P500, Nasdaq100, Russell 1000 & 2000 Return Series:2005-2014

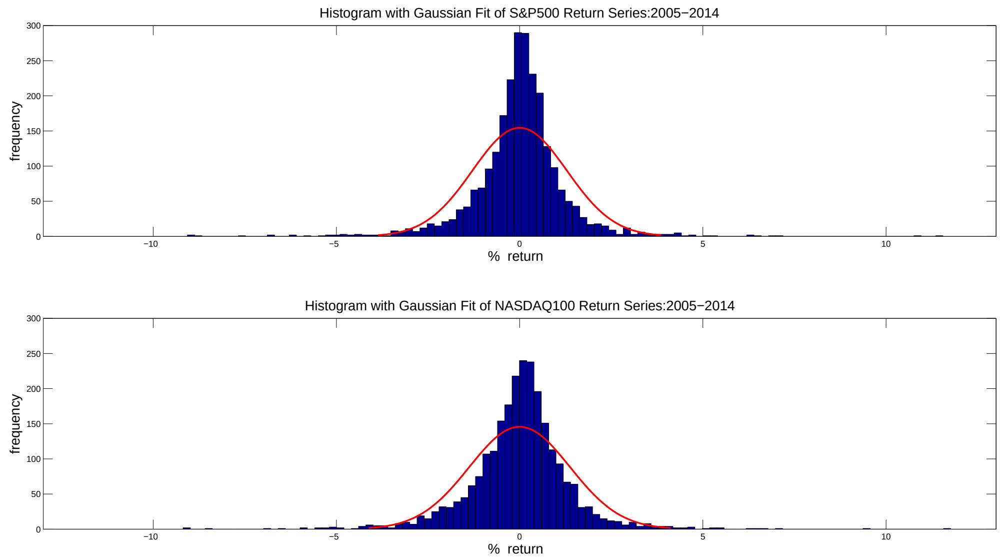

Figure 11: Histograms of S&P500 and NASDAQ100 Return series:2005-2014

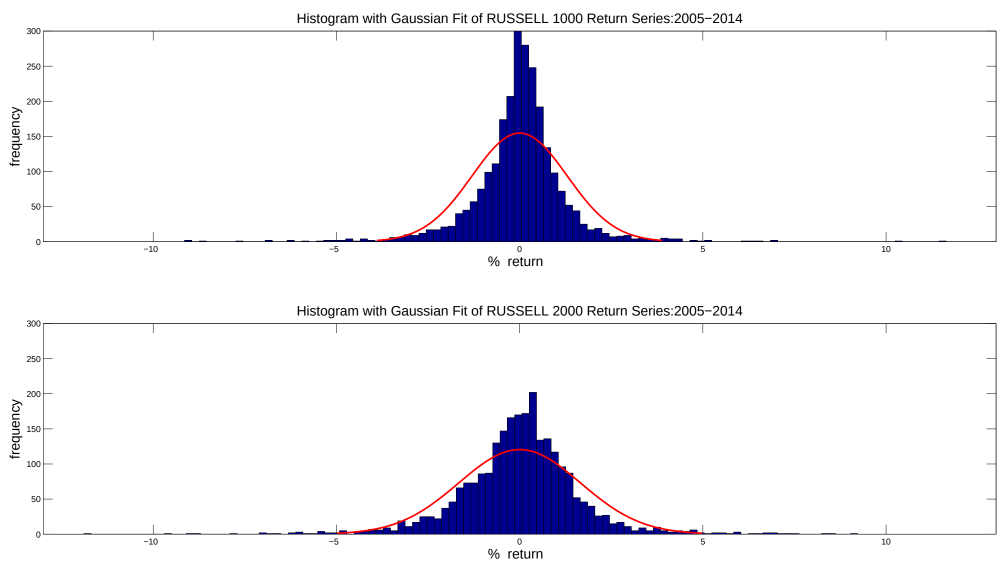

Figure 12: Histograms of Russell 1000 and Russell 2000 Return series:2005-2014

10. Applications in Risk Management: VaR and CVaR

Risk management play a critical role in the banks and financial institutions after the recent financial crisis in 2008-2009, and the central banks around the worlds are setting regulations on the risks exposed by the banks and financial institutions vs their reserve capital. The qualitative and quantitative methods are being applied to asses the risk. The Value-at-Risk (VaR) and Expected Shortfall/Conditional VaR (CVaR) are the two most important metrics has been as global standard for quantifying/measuring risk. The BASEL Accord is a global voluntary regulatory standard on risk for banks/financial institutions on capital adequacy, stress testing and market liquidity. Risk management is concerned with the tail of the predictive loss distribution, losses are defined as negative returns (negative returns are positive losses). The VaR metric uses the areas of *left tail* of return distribution. Also, by negating the returns the VaR can be computed from the *right tail* of the distribution. The SV model with the PF method has advantage to compute VaR dynamically with the sequential prediction/estimation of the volatility σ_t.

$$\mathrm{VaR}_\alpha = \sup[y \in \mathbb{R} : F_Y(y) \leq \alpha] \tag{19}$$

$$= \mu - \sigma_t z_\alpha \tag{20}$$

$$\text{Expected Shortfall, } \mathrm{ES}_\alpha = \mathbb{E}[Y | Y \leq \mathrm{VaR}_\alpha]$$

$$= \frac{1}{F_Y(\mathrm{VaR}_\alpha)} \int_{-\infty}^{\mathrm{VaR}_\alpha} y \, dF_Y(y) \tag{21}$$

where y_t is observed return, μ=expected return, α=1-c, c=confidence level, $\sigma_t = exp(x_t/2)$, x_t=log-volatility and $z_\alpha = \alpha$-quantile under Gaussian distributional assumption.

11. Conclusion

We applied PF method for sequential prediction/estimation of stochastic volatility modeled with leverage. The paper by Djuric et al. (2012) have shown that the leverage model outperforms all other existing models where no correlation in the noise processes are assumed. PF methods have shown to outperform for the non-linear/non-Gaussian systems, and the volatility is a non-linear process. Our method perform well in simulated data in figures (1)–(2). In application with real data, we estimated/predicted underlying volatility with the indexes of S&P500, NASDAQ 100, RUSSELL 1000 and RUSSELL 2000 on the daily closing prices for the year of 2005-2014. We plotted the simple returns and predicted log-volatility and volatility. We can't fit the volatility with the model, since actual volatility is unobserved, but, the graphical trend of the volatility looks quite good in association with observation series. The PF method on SV models can be applied for predicting/measuring risk dynamically, an especial application for HFT environment. The PF method is also suitable for developing HFT strategy. From the histograms figures of (11)–(12), we observe that return series are highly Kurtotic (peaked), thus severely deviates from the Gaussian distribution. We compute the sample *excess Kurtosis* from data series of S&P500, NASDAQ100, RUSSELL 1000 and RUSSELL 2000, and which are 11.27, 7.66, 10.96 and 5.11 respectively. Hence, Gaussian assumption will be a very poor approximation, since the excess Kurtosis for the Gaussian is 0. For improved modeling, Gaussian Mixtures Modeling (West, 1992; McLachlan and Peel, 2000) techniques should be applied.

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