We saw the ODE

$$y'' + \frac{y'}{x} = e^{y}$$

have 2 symmetries

$$X = c_1 x + c_2 x \ln x \qquad Y = -2c_1 - 2c_2(1 + \ln x)$$

Each one lead to a new ODE that was independent of s & with $u = s'$ lead to a first order ODE. A natural question is can one symmetry be used for the reduction and can the other symmetry be passed or "inherited" by the new ODE.

We define the Lie bracket as

$$[\Gamma_1, \Gamma_2] = \Gamma_1(\Gamma_2) - \Gamma_2(\Gamma_1)$$

As an example consider

$$\Gamma_1 = \frac{\partial}{\partial x} \quad \text{\&} \quad \Gamma_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \qquad \left(\begin{array}{c}\text{from}\\ \text{ex 2}\end{array}\right)$$

then
$$\Gamma_1(\Gamma_2(u)) = \frac{\partial}{\partial x}(x u_x - y u_y)$$
$$= u_x + x u_{xx} - y u_{xy}$$

$$\Gamma_2(\Gamma_1(u)) = \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)(u_x)$$
$$= x u_{xx} - y u_{xy}$$

so
$$[\Gamma_1, \Gamma_2](u) = u_x + x u_{xx} - y u_{xy} - (x u_{xx} - y u_{xy})$$
$$= u_x$$

so $[\Gamma_1, \Gamma_2] = \Gamma_1$

If Γ_i, Γ_j are 2 infinitesimal operators such that $[\Gamma_i, \Gamma_j] = k\Gamma_i$ $(k \in \mathbb{R})$

if we use Γ_i for the reduction, then the symmetry associated with Γ_j can be inherited by the new ODE,

Let us return to ex. 1

$$\Gamma_1 = x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} \qquad \Gamma_2 = x\ln x\frac{\partial}{\partial x} - 2(1+\ln x)\frac{\partial}{\partial y}$$

then $[\Gamma_1, \Gamma_2] = \Gamma_1$

so we use the symmetry associated Γ_1

and the symmetry associated with Γ_2 can be passed,

we already showed that

$$x = e^s, \quad y = \ln r - 2s \qquad (\text{C of V})$$

lead to

$$r s'' + s' + r^3 s'^3 = 0$$

and if $s' = u$, $s'' = u'$

giving $\quad r u' + u + r^3 u^3 = 0 \qquad - (1)$

so how do we pass

$$X_2 = x \ln x, \quad Y_2 = -2(1 + \ln x) \qquad (\Gamma^2)$$

to the new ODE $-(1)$

From CofV

$$r = x^2 e^y, \quad s = \ln x$$

then $\Gamma^* = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + R \frac{\partial}{\partial r} + S \frac{\partial}{\partial s}$

so $\quad R = 2x X e^y + x^2 e^y Y$

$\quad S = \frac{1}{x} X$

from Γ^2 & Cof r

$$R = 2xe^y \dot{x}\ln x + x^2e^y\left(-2(1+\ln x)\right)$$

$$= \cancel{2x^2\ln x e^y} - 2x^2e^y - \cancel{2x^2\ln x e^y}$$

$$= -2x^2e^y = -2r$$

& $S = \frac{1}{x}X = \frac{1}{x}x\ln x = \ln x = s$

so $R = -2r,\ S = s$

& the 1^{st} extension

$$S_{(r)} = S_r + (S_s - R_r)s' - R_s s'^2$$

$$= 0 + (1-(-2))s' - 0$$

$$= 3s'$$

Now $s' = u$ so $S[r] = \overline{U}$

$\Rightarrow \quad \overline{U} = 3s' = 3u$

so the new symmetry for

$$ru' + u + r^3u^3 = 0 \qquad (*)$$

is $\Gamma = -2r\frac{\partial}{\partial r} + 3u\frac{\partial}{\partial u}$

and you can check Lie's invariance condition that this is a symmetry for $*$)

<u>Ex 2</u> $\quad y'' + yy' + y^3 = 0$

Admits

$$\Gamma_1 = \frac{\partial}{\partial x} \qquad \Gamma_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

$[\Gamma_1, \Gamma_2] = \Gamma_1$ so use Γ_1 & pass Γ_2

C of V

$$r_x = 0 \qquad s_x = 1$$

$$\Rightarrow \quad r = R(y) \qquad s = x + S'(y)$$

choose $r = y,\ s = x$ or $x = s,\ y = r$

$$y' = \frac{1}{s'}, \quad y'' = -\frac{s''}{s'^3}$$

so $y'' + yy' + y^3 = 0 \Rightarrow -\dfrac{s''}{s'^3} + \dfrac{r}{s'} + r^3 = 0$

or $\quad -s'' + rs'^2 + r^3 s'^3 = 0$

or $\quad s'' - rs'^2 - r^3 s'^3 = 0$

let $s' = u$ so $u' - ru^2 - r^3u^3 = 0$

if $r = y \Rightarrow R = T = -y = -r$

$s = x \Rightarrow S = X = x = s$

Now $S_{[r]} = S_r + (S_s - R_r)s' - R_s s'^2$

$= 0 + (1-(-1))s' - 0$

$= 2s'$

so $s' = u \Rightarrow S_{[r]} = U = 2s' = 2u$

so $R = -r,\ U = 2u$

$\therefore$ $\Gamma = -r\frac{\partial}{\partial r} + 2u\frac{\partial}{\partial u}$

is a sym. of $u' - ru^2 - r^3u^3 = 0$