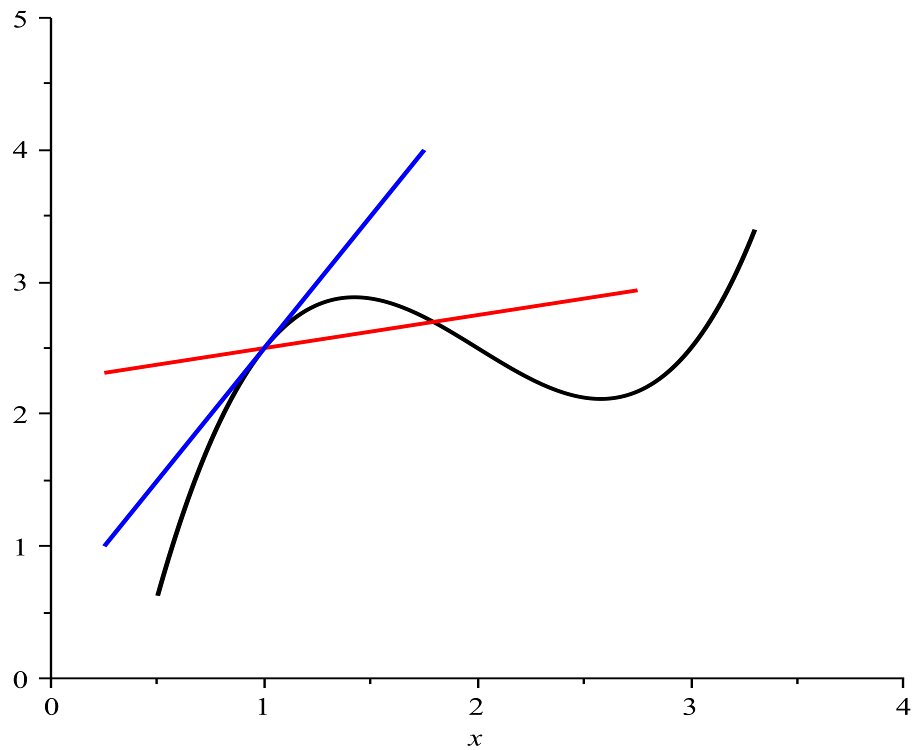
Calculus 3 - Vector Functions

Derivatives

We considered the function $y = f(x)$ and a secant to the curve that goes through the points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$ (in red). Then we let $\Delta x \to 0$ and the secant line (red) becomes the tangent line (blue)



Mathematically, we define the derivative of a function $y = f(x)$ as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1}$$

We do the same thing for vector functions.

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \tag{2}$$

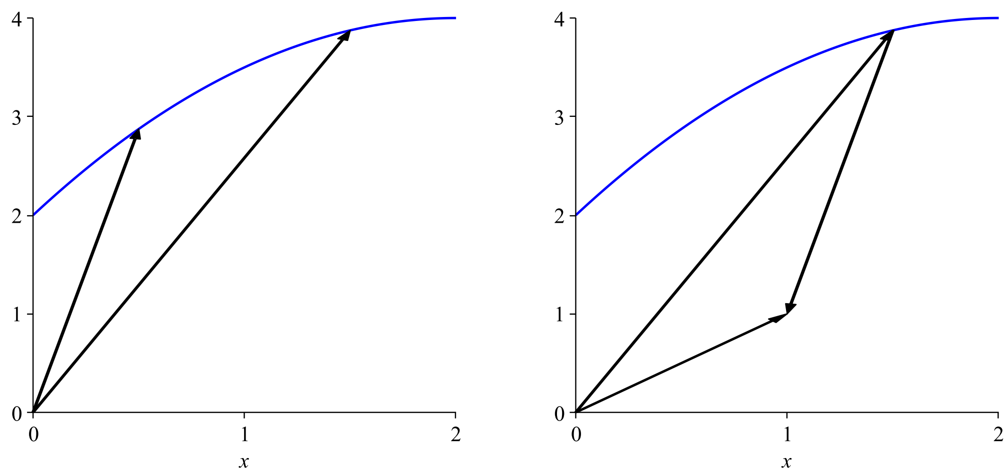


Figure 1: Vector Functions $\overrightarrow{r}(t)$, $\overrightarrow{r}(t+\Delta t)$ and $\overrightarrow{r}(t+\Delta t)-\overrightarrow{r}(t)$

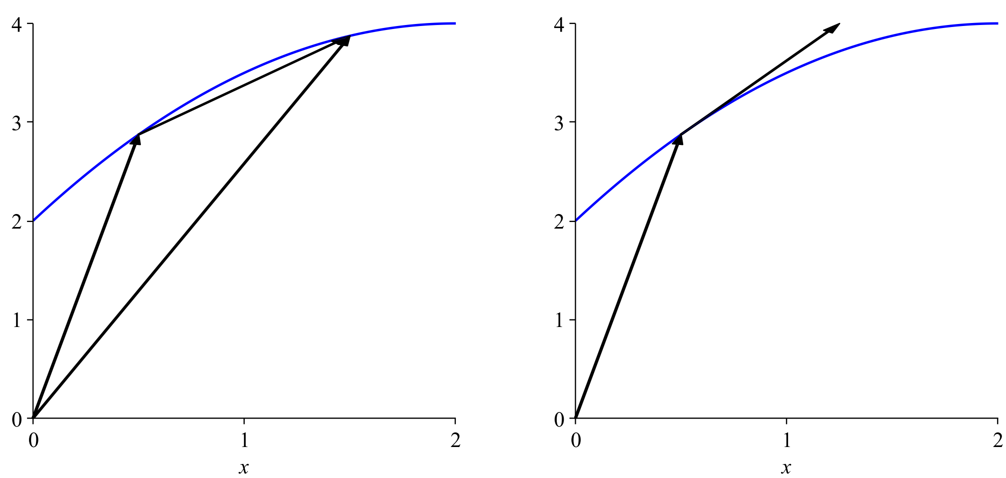


Figure 2: $\lim\limits_{\Delta t \to 0} \dfrac{\overrightarrow{r}(t+\Delta t)-\overrightarrow{r}(t)}{\Delta t}$

Now we consider the derivative analytically for the vector function $\vec{r}(t) =< f(t), g(t) >$.

$$
\begin{aligned}
\vec{r}'(t) &= \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{< f(t+\Delta t), g(t+\Delta t) > - < f(t), g(t) >}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t} \right\rangle \\
&= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} \right\rangle \\
&= < f'(t), g'(t) >
\end{aligned}
\tag{3}
$$

so we have

$$
\vec{r}'(t) =< f'(t), g'(t) > \tag{4}
$$

and all the derivative rules from Calc 1 apply.

Example 1

If $\vec{r}(t) =< t, t^2 >$ find $\vec{r}'(t)$.

A simple calculation gives $\vec{r}'(t) =< 1, 2t >$

Example 2

If $\vec{r}(t) =< \cos t, \sin t >$ find $\vec{r}'(t)$.

A simple calculation gives $\vec{r}'(t) =< -\sin t, \cos t >$

Example 3

If $\vec{r}(t) =< t^2 - 1, 2t, t^2 + 1 >$ find $\vec{r}'(t)$ and $\vec{r}''(t)$.

A simple calculation gives $\vec{r}'(t) =< 2t, 2, 2t >$ and $\vec{r}''(t) =< 2, 0, 2 >$. A physical interpretation of $\vec{r}''(t)$ will be given later.

Integrals

As much as we can take derivatives of vector functions, we can integrate vector functions. So if $\vec{r}(t) =< f(t), g(t) >$ then for indefinite integrals

$$\int \vec{r}(t)dt =< \int f(t)dt + c_1, \int g(t)dt + c_2 > \tag{5}$$

where c_1 and c_2 are arbitrary constants and for definite integrals

$$\int_a^b \vec{r}(t)dt =< \int_a^b f(t)dt, \int_a^b g(t)dt > . \tag{6}$$

Example 4
If $\vec{r}(t) =< \cos t, \sin t >$ find $\int \vec{r}(t)dt$.
A simple calculation gives
$\int \vec{r}(t)dt =< \int \cos t dt + c_1, \int \sin t dt + c_2 >=< \sin t + c_1, -\cos t + c_2 >$

Example 5
If $\vec{r}(t) =< 3t^2, 2t, \frac{1}{t^2} >$ find $\int_1^2 \vec{r}(t)dt$.
A simple calculation gives

$$\begin{aligned} \int_1^2 \vec{r}(t)dt &= \left\langle \int_1^2 3t^2 dt, \int_1^2 2t dt, \int_1^2 \frac{1}{t^2} dt \right\rangle \\ &= \left\langle t^3 \Big|_1^2, \ t^2 \Big|_1^2, \ -\frac{1}{t} \Big|_1^2 \right\rangle \\ &= \left\langle 7, 3, \frac{1}{2} \right\rangle \end{aligned} \tag{7}$$