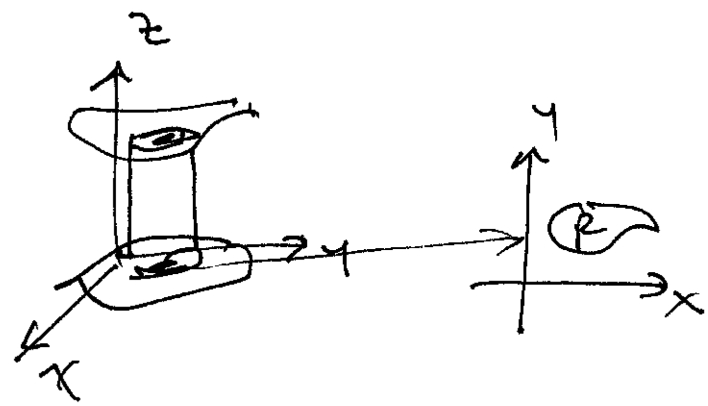
Math 2471 - Calc 3

Triple integrals

$$\iiint_V f(x,y,z)\,dV$$

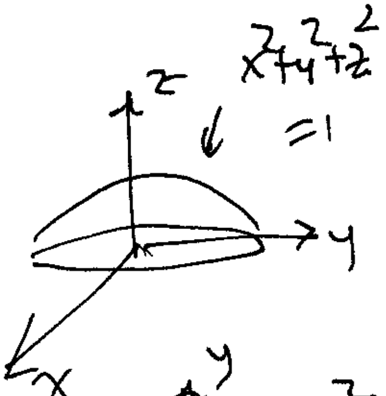


Next - Triple in cylindrical polars

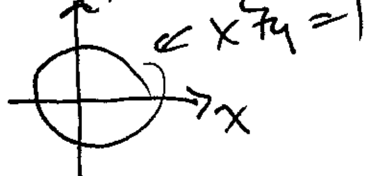
$x = r\cos\theta,\ y = r\sin\theta \quad dV = r\,dz\,dr\,d\theta$

Really good if we find volumes that involve cylinders

Left off



$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{\sqrt{1-r^2}} 1\; r\,dz\,dr\,d\theta$$

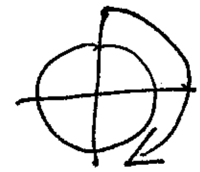


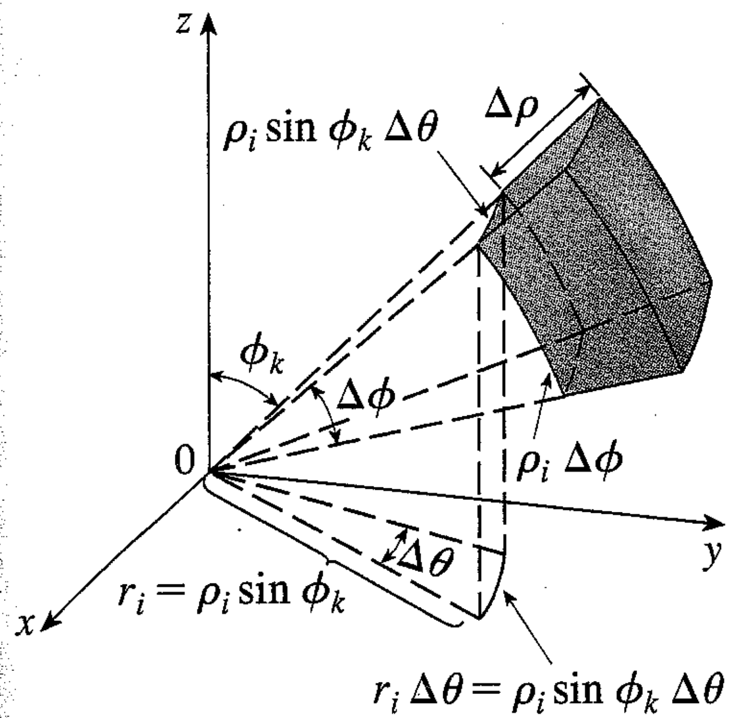
better involving spherical polars

Now $x = \rho\cos\theta\sin\phi$ $\quad 0 \le \theta \le 2\pi$

$y = \rho\sin\theta\sin\phi$ $\quad 0 \le \phi \le \pi$

$z = \rho\cos\phi$





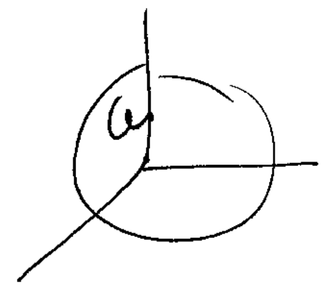
so

$$dV = \rho\sin\phi\, d\theta \cdot \rho\, d\phi \cdot d\rho$$

$$= \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

For limits of integration we see what ρ, θ, & ϕ do to sweep out the volume!

ex1 $\iiint_V 1\,dV$ where V is the sphere of radius 1



$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

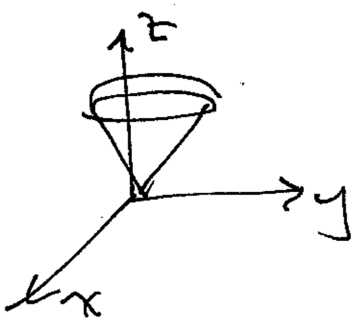
$$=\int_0^{2\pi}\int_0^{\pi}\frac{\rho^3}{3}\Big|_0^1\sin\phi\,d\phi\,d\theta=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}-\cos\phi\Big|_0^{\pi}d\theta=\frac{1}{3}\int_0^{2\pi}\underbrace{(-\cos\pi+\cos 0)}_{1/2}\,d\theta$$

$$=\frac{2}{3}\cdot\int_0^{2\pi}d\theta=\frac{2}{3}\theta\Big|_0^{2\pi}=\frac{4\pi}{3}$$

$V_{sp}=\frac{4}{3}\pi r^3$ so when $r=1$ $V=\frac{4\pi}{3}$ ✓

Ex2 Find the volume inside the sphere $x^2+y^2+z^2=2$ & the cone $z=\sqrt{x^2+y^2}$



$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{\sqrt{2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

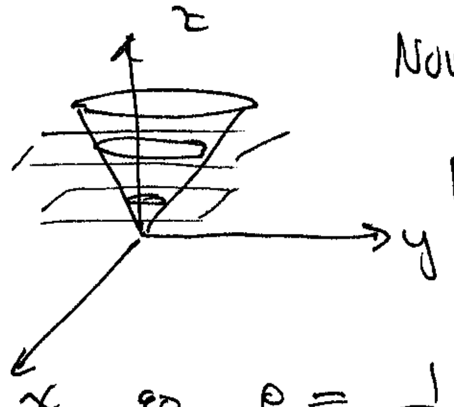
$$=\int_0^{2\pi}\int_0^{\pi/4}\frac{\rho^3}{3}\Big|_0^{\sqrt{2}}\sin\phi\,d\phi\,d\theta = \frac{2\sqrt{2}}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\,d\phi\,d\theta$$

$$=\frac{2\sqrt{2}}{3}\int_0^{2\pi}-\cos\phi\Big|_0^{\pi/4}d\theta = \frac{2\sqrt{2}}{3}\int_0^{2\pi}\left(-\cos\frac{\pi}{4}+1\right)d\theta$$

$$=\frac{2\sqrt{2}}{3}\left(1-\frac{\sqrt{2}}{2}\right)\theta\Big|_0^{2\pi} = \frac{4\sqrt{2}\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)$$

#52 inside cone $z = \sqrt{x^2+y^2}$ between

the plane $z=1$ & $z=2$



Now $\phi: 0 \to \pi/4$ (like last quest)

But ρ goes from $z=1 \to z=2$

$z = \rho\cos\phi$

so $\rho = \frac{1}{\cos\phi} \to \frac{2}{\cos\phi}$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\sec\phi}^{2\sec\phi} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\sin\phi\Big|_{\frac{1}{\cos\phi}}^{\frac{2}{\cos\phi}} d\phi \, d\theta = \int_0^{2\pi} \frac{7\sin\phi}{3\cos^3\phi} \, d\phi \, d\theta$$

$$= \int_0^{2\pi} + \frac{7}{3\cdot 2\cos^2\phi}\Big|_0^{\pi/4} d\theta = \int_0^{2\pi} + \frac{7}{3\cdot 2}\left(\frac{1}{\cos^2\pi/4} - 1\right) d\theta$$

$$= \int_0^{2\pi} + \frac{7}{3\cdot 2}(2-1)\,d\theta = \frac{7}{2\cdot 3}\int_0^{2\pi} d\theta = \frac{7\theta}{2\cdot 3}\Big|_0^{2\pi} = \cancel{14\pi}$$

$$= 7\pi/3$$