Math 6345 – AODEs

Bifurcations

$$\dot{x} = F(x, \lambda) \qquad \lambda \in \mathbb{R}$$

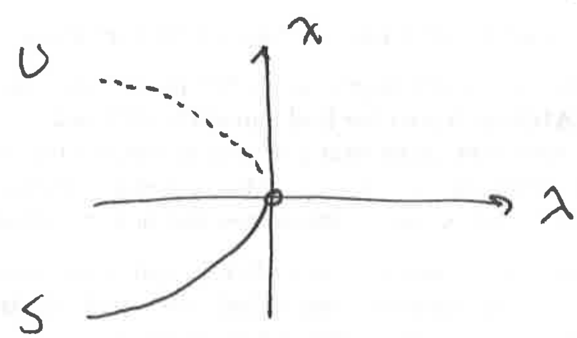
we considered 3 types

(1) Saddle node

$$\dot{x} = x^2 + \lambda$$

critical pts depend on λ: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$

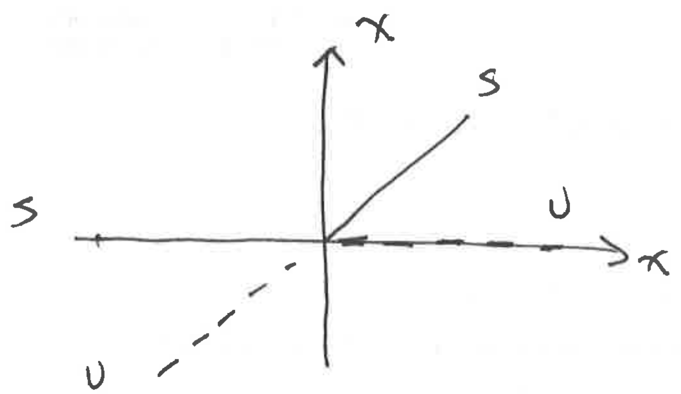
Bifurcation Diagram



(2) Transcritical

$$\dot{x} = \lambda x - x^2 \qquad \text{CP } x = 0,\ x = \lambda$$
$$= x(\lambda - x)$$

but the CP $x = 0$ changes it stability as λ changes

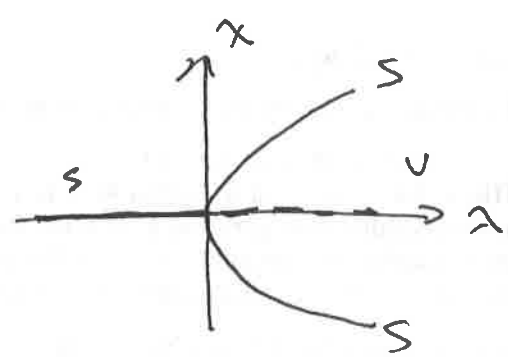


3) <u>Pitchfork</u>

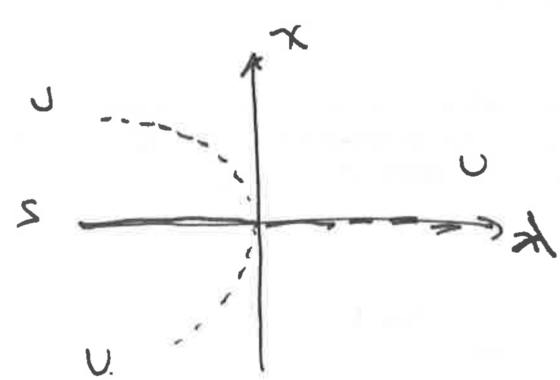
$$\dot{x} = \lambda x - x^3$$
$$= x(\lambda - x^2)$$

CP $x=0$ persists
and other CPs (2 of them)
depends on λ

Supercritical



subcritical $\quad \dot{x} = \lambda x + x^3$
$$= x(\lambda + x^2)$$



In each of the 3 ~~scens~~ Scenarios:
the bifurcation pt was $(x, \lambda) = (0,0)$

so we define this as (x_c, λ_c) c – critical pt

So given $\dot{x} = F(x, \lambda)$

Can we determine the type of bifurcation

(1) saddle-node

(2) transcritical

(3) pitch fork

Let's reconsider

$$\left.\begin{aligned} \dot{x} &= \lambda x + x^2 = F(x,\lambda) \\ \dot{x} &= \lambda x - x^2 = F(x,\lambda) \\ \dot{x} &= \lambda x - x^3 = F(x,\lambda) \end{aligned}\right\} \quad \begin{aligned} x_c &= 0 \\ \lambda_c &= 0 \end{aligned}$$

In each case

$F_x = 2x,$ $\quad F_x = \lambda - 2x$ $\quad F_x = \lambda - 3x^2$

and at the bifurcation pt

$F_x = 0$ non hyperbolic

so if the bifurcation will look like one of our 3 prototypes then we require

$$F(x_c, \lambda_c) = 0$$
$$F_x(x_c, \lambda_c) = 0$$

Taylor Series

$$\dot{x} = \overset{0}{\cancel{F(x_c, \lambda_c)}} + \overset{0}{\cancel{F_x(x_c, \lambda_c)}}(x - x_c)$$
$$+ F_\lambda(x_c, \lambda_c)(\lambda - \lambda_c)$$
$$+ F_{xx}(x_c, \lambda_c)(x - x_c)^2/2!$$
$$+ F_{x\lambda}(x_c, \lambda_c)(x - x_c)(\lambda - \lambda_c)$$
$$+ F_{\lambda\lambda}(x_c, \lambda_c)\frac{(\lambda - \lambda_c)^2}{2!}$$

then translate Bif pt to $(0,0)$

Ex $\qquad \dot{x} = \lambda - x - e^{-x}$

So here $\qquad F(x,\lambda) = \lambda - x - e^{-x}$

$$F_x(x,\lambda) = -1 + e^{-x}$$

Set each to zero

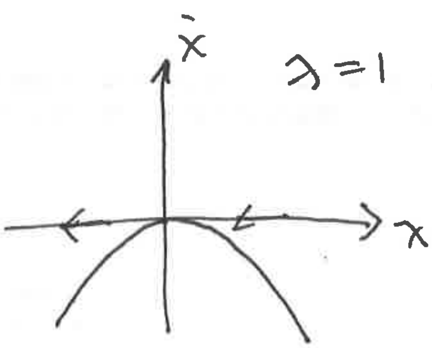
so $\qquad -1 + e^{-x_c} = 0 \qquad \lambda_c - x_c - e^{-x_c} = 0$

$$\Rightarrow \quad (x_c, \lambda_c) = (0, 1)$$

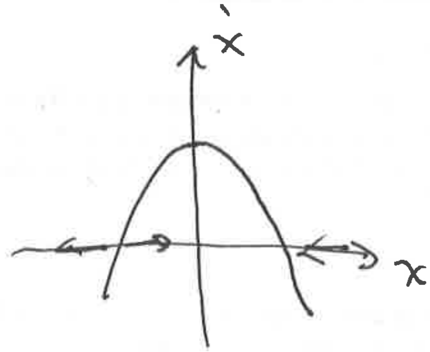
Now $\qquad F_\lambda = 1$

$$F_{xx} = -e^{-x} \qquad F_{x\lambda} = 0 \qquad F_{\lambda\lambda} = 0$$

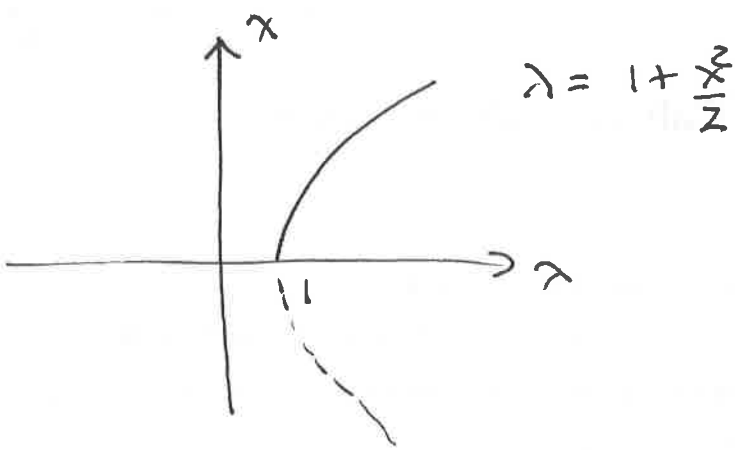
so $\qquad \dot{x} = 1(\lambda - 1) - \frac{x^2}{2} + O(x^3)$ $\qquad$ Saddle node



$\lambda = 1$



$\lambda = 2$



See Maple plots for a comparison

Ex 2
$$\dot{x} = \lambda x - \frac{x}{x+1}$$

so here $$F(x, \lambda) = \lambda x - \frac{x}{x+1}$$

$$F_x = \lambda - \frac{1}{(1+x)^2}$$

Set each to zero

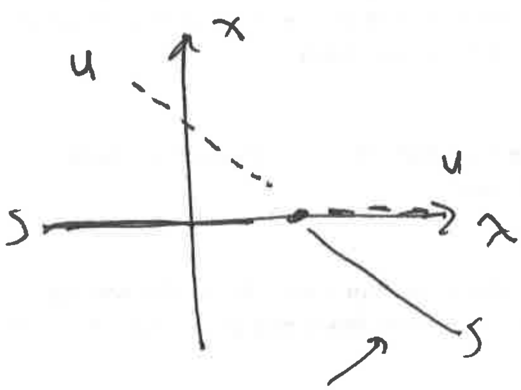
$$x_c\left(\lambda_c - \frac{1}{x_c+1}\right) = 0 \qquad \lambda_c - \frac{1}{(1+x_c)^2} = 0$$

CP $\quad x_c = 0 \quad \lambda_c = 1$

Taylor Expansion

$$\dot{x} = x(\lambda - 1 + x) + O(x^3) \qquad \text{transcritical}$$

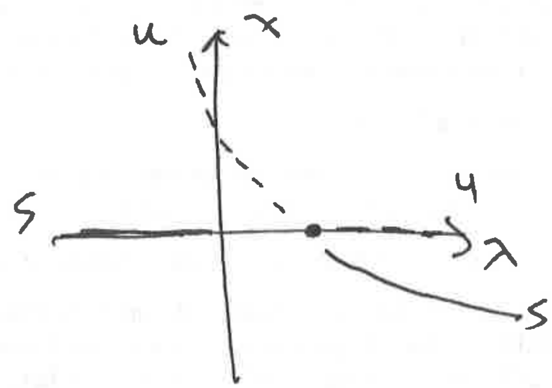
Approx



$$\lambda = 1 - x$$

or $x = 1 - \lambda$

Exact



$$\lambda = \frac{1}{x+1} \quad \text{or} \quad x = \frac{1}{\lambda} - 1$$

Ex3 $\quad \dot{x} = \lambda x - \dfrac{x}{1+x^2}$

Critical/Bif. Pt is again $(x_c, \lambda_c) = (0, 1)$

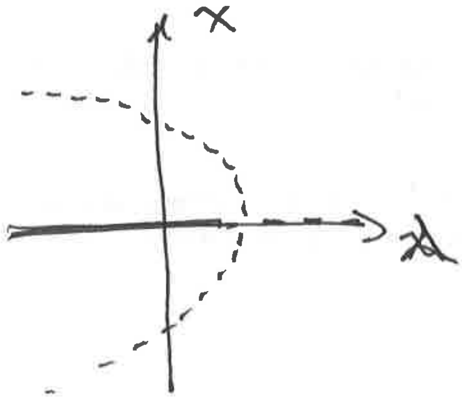
~~Taylor expansion~~ $\quad F = \lambda x - \dfrac{x}{1+x^2}$

~~$\dot{x} =$~~ $\quad F_x = \lambda + \dfrac{x^2 - 1}{(1+x^2)^2}$

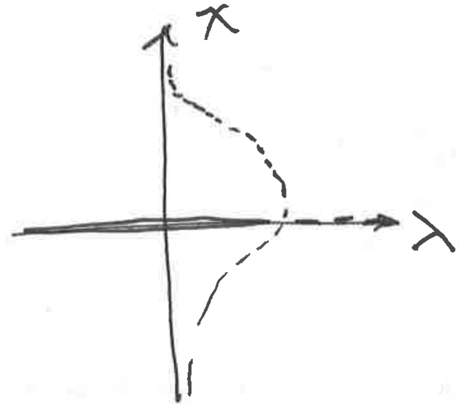
Taylor Expansion

$$\dot{x} = (\lambda-1)x + x^3 \qquad \text{Pitchfork Bif.}$$

Approx



Exact



$$\lambda = \frac{1}{1+x^2}$$