On the Trade-off Between Controllability and Robustness in Networks of Diffusively Coupled Agents

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***Abstract*— In this paper, we study the relationship between two crucial properties in linear dynamical networks of diffusively coupled agents, that is controllability and robustness to noise and structural changes in the network. In particular, for any given network size and diameter, we identify networks that are maximally robust and then analyze their strong structural controllability. We do so by determining the minimum number of leaders to make such networks completely controllable with arbitrary coupling weights between agents. Similarly, we design networks with the same given parameters that are completely controllable independent of coupling weights through a minimum number of leaders, and then also analyze their robustness. We utilize the notion of Kirchhoff index to measure network robustness to noise and structural changes. Our controllability analysis is based on novel graph-theoretic methods that offer insights on the important connection between network robustness and strong structural controllability in such networks.**

I. Introduction

In a networked control system, controllability and robustness to noise and structural changes in the network are two of the most crucial attributes. Controllability describes the ability to manipulate and drive the network to a desired state through external inputs, whereas, network robustness expresses the ability of the network to maintain its structure in the event of device or link failures. Exploiting trade-offs between network controllability and robustness can have a far reaching impact on the overall network design.

In this paper, we study the relationship between controllability and robustness in diffusively coupled leader-follower networks by focusing on finding extremal networks for these properties. In particular, for given parameters, we obtain networks with maximal robustness and then analyze their controllability. Similarly, we design networks with maximal controllability, and then evaluate their robustness. To characterize network robustness, we utilize a widely used metric *Kirchhoff index* (K_f) that captures both aspects of robustness, that is, the effect of structural changes in the network as well as the effect of noise on the overall dynamics (for instance, see [1], [2], [3]). To quantify control performance, we consider the minimum number of inputs (leaders) needed to make the network *strong structurally controllable*, that is, completely controllable irrespective of the coupling weights between nodes (e.g., see [4], [5], [6]). Accordingly, a network that requires fewer leaders for strong structural controllability is preferred over the one requiring many leaders.

Our approach is primarily graph-theoretic, and turns out to be effective in exploiting the relationship between network controllability and robustness. Our main contributions are:

- For any given number of nodes N and diameter D, we identify networks with maximum robustness and provide a detailed analysis of their controllability, that is, the number of leaders that are necessary and sufficient to completely control such networks with arbitrary coupling weights between nodes.
- For any number of nodes N and diameter D, we design networks that are strong structurally controllable with the minimum number of leaders. For this, we first provide a sharp upper bound on the minimum number of leaders for strong structural controllability with arbitrary N and D.
- We also evaluate the robustness of maximally controllable networks and compare it with the robustness of maximally robust graphs for the same N and D.

Kirchhoff index or equivalently effective graph resistance based measures have been useful in quantifying the effect of noise on the expected steady state dispersion in linear dynamical networks, [1], [7], [8]. To maximize robustness by minimizing Kirchhoff index, various optimization approaches have been proposed (e.g., [3], [9]). The main objective there is to determine crucial edges that need to be added or maintained to maximize robustness under given constraints [10]. To quantify controllability, several approaches have been adapted, such as determining the minimum number of inputs (leader nodes) needed to (structurally or strong structurally) control a network, determining the worst-case control energy based on controllability Gramians, and so on (e.g., see [11], [12]). Strong structural controllability, due to its independence on coupling weights between nodes, is a generalized notion of controllability. There are various studies providing graph-theoretic characterizations of this concept [4], [5], [6].

Very recently in [13], trade-off between controllability and fragility in complex networks is investigated. Fragility measures the smallest perturbation in edge weights to make the network unstable. Authors in [13] show that networks that require small control energy to drive from one state to another, as measured by the eigen values of the controllability Gramian, are more fragile and vice versa. In our work, for control performance, we consider minimum leaders for strong structural controllability, which is independent

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of coupling weights; and for robustness, we utilize the Kirchhoff index which measures robustness to noise as well as to structural changes in the underlying network graph. Moreover, in this work we focus on designing and comparing extremal networks for these properties.

II. PRELIMINARIES

Let $\mathcal{G}(\mathcal{V},\mathcal{E})$ be an undirected graph with a vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. The graphs in this paper are loop-free, that is, no self loops between nodes. A node u is a neighbor of v if an edge exists between u and v, which is denoted by an unordered pair (u,v). The *neighborhood* of u is denoted by $\mathcal{N}_u = \{v \in \mathcal{V} | (u,v) \in \mathcal{E}\}$. The *distance* between nodes u and v, denoted by $d(u,v)$, is the number of edges in the shortest path between u and v. The *diameter* of $\mathcal{G}$, denoted by $\mathcal{D}$, is the maximum distance between any two nodes in $\mathcal{G}$. A graph is *weighted* if edges are assigned values (weights) using some weighting function $w : \mathcal{E} \longrightarrow \mathbb{R}_+$. The *adjacency* matrix of $\mathcal{G}$ is defined as

$$\mathcal{A}_{ij} = \begin{cases} w(i,j) & \text{if } (i,j) \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Similarly, the *degree* matrix of $\mathcal{G}$ is defined as

$$\Delta_{ij} = \begin{cases} \sum_{k \in \mathcal{N}_i} \mathcal{A}_{ik} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The *Laplacian* of $\mathcal{G}$ is then defined as

$$\mathcal{L} = \Delta - \mathcal{A}. \tag{3}$$

A. Network Dynamics

We consider a network of agents modeled by a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ in which the node set $\mathcal{V} = \{1, 2, \dots, N\}$ represents agents and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents interconnections between agents. Each agent i updates its state $x_i \in \mathbb{R}$ by the following dynamics

$$\dot{x}_i(t) = -\sum_{j \in \mathcal{N}_i} w(i,j)(x_i(t) - x_j(t)), \tag{4}$$

where $w(i,j)$ is the coupling strength between nodes i and j. Moreover, to control and drive the network as desired, external control inputs are injected through a subset of nodes called *leaders*. The dynamics of the leader node i is,

$$\dot{x}_i(t) = -\sum_{j \in \mathcal{N}_i} w(i,j)(x_i(t) - x_j(t)) + u_i(t). \tag{5}$$

Let the set of leaders be represented as $\mathcal{V}_L = \{\ell_1, \dots, \ell_k\} \subseteq \mathcal{V}$, where, without loss of generality, the leaders are labeled such that $\ell_j < \ell_{j+1}$. If the total number of nodes is N and the number of leader nodes is k, then the overall system level dynamics can be written using the underlying graph's Laplacian as

$$\dot{x}(t) = -\mathcal{L}x(t) + \mathcal{B}u(t), \tag{6}$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_N(t) \end{bmatrix}^T \in \mathbb{R}^N$ be the state vector, $u(t) \in \mathbb{R}^k$ be the control input to the leaders, and $\mathcal{B}$ be an $N \times k$ input matrix with the following entries

$$\mathcal{B}_{ij} = \begin{cases} 1 & \text{if } i = \ell_j \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

III. NETWORK MEASURES AND PROBLEM SETUP

A. Robustness Measure

To measure network robustness, we use the notion of *Kirchhoff index* of a graph, denoted by K_f, and defined as

$$K_f = N \sum_{i=2}^{N} \frac{1}{\lambda_i}, \tag{8}$$

where N is the number of nodes and $\lambda_2 \leq \lambda_3 \leq \cdots \lambda_N$ are positive eigenvalues of the Laplacian of the graph (weighted or unweighted). A smaller value of K_f indicates higher robustness in networks and vice versa.

Our motivation to use this robustness measure is twofold. First, it is very useful in characterizing the robustness to noise of linear consensus over networks. In fact, as shown in [1], it is directly related to the H_2 norm that measures the expected steady-state dispersion of the nodes under white noise via the relationship $H_2 = \left(\frac{K_f}{2N}\right)^{\frac{1}{2}}$. Thus, it characterizes the *functional robustness* – ability of the network to perform well in the presence of noise that corrupts measurements or information exchange within the network. Other applications of K_f in the study of various control theoretic problems have been surveyed in [7].

Second, K_f of a network captures its *structural robustness* – the ability of the network to retain its structural attributes in the case of edge or node deletions. It assimilates the effect of not only the number of paths between nodes, but also their quality as determined by the lengths of the paths [3]. For a detailed discussion, we refer the readers to [2], [3], [9].

B. Controllability Measure

A state $x \in \mathbb{R}^N$ is reachable if there exists an input that can drive the system in (6) from origin to x in finite time. A set of all reachable states constitutes the *controllable subspace*, which is the range space of the following matrix.

$$\Gamma = \begin{bmatrix} \mathcal{B} & -\mathcal{L}\mathcal{B} & (-\mathcal{L})^2\mathcal{B} & \cdots & (-\mathcal{L})^{N-1}\mathcal{B} \end{bmatrix} \tag{9}$$

The dimension of controllable subspace is the rank of Γ, which needs to be N for complete controllability. The rank of Γ depends not only on the edge set of the graph but also on the edge weights. In fact, a graph that is completely controllable for one set of edge weights might not remain completely controllable if edge weights are changed. For a given graph and leader nodes (inputs), the minimum rank of Γ for any choice of edge weights is the *dimension of strong structurally controllable* subspace. A graph is said to be *strong structurally controllable* with a given set of leaders, if the resulting controllability matrix Γ is full rank with *any* choice of edge weights. Thus, in a strong structurally controllable network, perturbation in edge weights has no effect on the dimension of controllable subspace, which makes the notion of strong structural controllability quite general and applicable in situations where exact information of edge weights is inscrutable.

As a result, we are interested in finding the minimum number of leaders required to make a network *strong structurally controllable*.

C. Problems

We are interested in exploring relationships and trade-offs between robustness and controllability (as defined above) in diffusively coupled systems (6). In particular, we focus on extremal cases, and look at the following problems.

1. For given number of nodes N and diameter D, which graphs have the minimum K_f and thus, the maximum robustness?
2. What is the control performance, as measured by the minimum number of leaders needed for strong structural contrallability, of the maximally robust graphs?
3. For any N and D, what is the minimum number of leaders that guarantee strong structural controllability? Furthermore, how can we construct graphs that achieve strong structural controllability with that many leaders.
4. What is the robustness of graphs in point (3) above?

IV. Maximally Robust Networks and their Controllability

In this section, our goal is to identify maximally robust networks, and then analyze their controllability.

A. Maximally Robust Networks

For a given N and D, which graphs are maximally robust, that is, have the minimum K_f amongst all such graphs? Another way to state this problem is to consider a complete graph of N nodes, denoted by $\mathcal{K}_N$, and obtain a subgraph of $\mathcal{K}_N$ that has a diameter D and has the minimum K_f amongst all such subgraphs.

For the unweighted case, it has been shown explicitly in [3] that for any N and D, optimal graphs having the minimum K_f belong to a special class known as the *clique chains*, defined below. A clique is a subgraph in which all vertices are pairwise adjacent.

Definition (*Clique chain [3]*) Let $n_1, n_2, \cdots, n_D, n_{D+1}$ be a set of positive integers and $N = \sum_{i=1}^{D+1} n_i$, then a clique chain of N nodes and diameter D is a graph obtained from a path graph of diameter D, that is P_{D+1}, by replacing each node with a clique of size n_i such that the vertices in distinct cliques are adjacent if and only if the corresponding original vertices in the path graph are adjacent. We denote such a clique chain by $\mathcal{G}_D(n_1, \cdots, n_{D+1})$.

An example is illustrated in Figure 1.

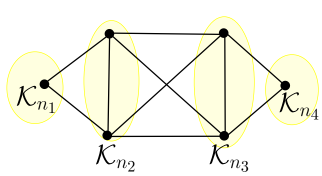

Fig. 1: $\mathcal{G}_3(1, 2, 2, 1)$ – A clique chain with 6 nodes and diameter 3 with $n_1 = 1, n_2 = 2, n_3 = 2$, and $n_4 = 1$.

In fact, the following result establishes optimality of clique chains in terms of the minimum K_f.

Theorem 4.1: [3] For a given number of nodes N and D, graphs that achieve the minimum K_f are necessarily clique chains of the form $\mathcal{G}_D(n_1 = 1, n_2, \cdots, n_D, n_{D+1} = 1)$ where $N = \sum_{i=1}^{D+1} n_i$.

Note that the n_1 and n_{D+1} are always 1 in the optimal clique chains. For the weighted case, assume that $\mathcal{K}_N$ is a complete graph with edge weights, and the question is to obtain a weighted spanning subgraph of $\mathcal{K}_N$ that has a diameter D and has the minimum K_f. Using the same arguments as in [3], we state the following:

Proposition 4.2: If $\mathcal{K}_N$ is a weighted complete graph, then among all the subgraphs of $\mathcal{K}_N$ with N nodes and diameter D, the graph that has the minimum K_f is a clique chain $\mathcal{G}_D(1, n_2, \cdots, n_D, 1)$ where $\sum_{i=1}^{D+1} n_i = N$.

Thus, for a given N and D, maximally robust graphs (both for the weighted and unweighted cases) are clique chains of the form $G_D(1, n_2, \cdots, n_D, 1)$.

B. Controllability of Clique Chains

Next, we analyze the strong structural controllability of the maximally robust graphs, that is, clique chains. The main result of this section is stated below.

Theorem 4.3: Let $\mathcal{G}_D(n_1, \cdots, n_{D+1})$ be a clique chain with diameter $D > 2$, and k be the number of leaders needed for the strong structural controllability of $\mathcal{G}_D$, then

$$N - (D + 1) \leq k \leq N - D. \qquad (10)$$

We prove this result in [14] using the following graph-theoretic notions:

- The *maximal leader invariant external equitable partitions (LIEEP)* [15], [16] to get the lower bound, and
- the notion of distance-to-leaders vectors and *pseudo-monotonically increasing sequences (PMI)* that we introduced in [5] to get the upper bound. We explain these concepts with an example in Appendix.

To obtain the lower bound in (10), we first note that the maximal LIEEP consisting of only singleton cells is a necessary condition for complete controllability. Next, we determine the minimum number of leaders to have such a maximal LIEEP, which directly gives the minimum number of leaders for strong structural controllability. For the upper bound in (10), we determine the minimum number of leaders such that the graph has a *full PMI sequence* (see Appendix), which in turn would imply that the network is strong structurally controllable with that many leaders. A detailed proof is available in [14].

V. Maximally Controllable Networks and their Robustness

In the previous section, we looked at maximally robust networks, and analyzed their controllability. Here, we obtain graphs that are strong structurally controllable with the minimum leaders and evaluate their robustness.

A. Maximally Controllable Networks

For any given N and D, which graphs exhibit strong structural controllability with the minimum number of leaders? To answer this, we first need to study for an arbitrary N and D, what is the minimum number of leaders needed to guarantee strong structural controllability? One of the main results in this section is as follows:

Theorem 5.1: For any N and D, there exist graphs that are strong structurally controllable with k leaders, where

$$k \leq \left\lceil \frac{N-1}{D} \right\rceil . \tag{11}$$

Remark 1 - The above bound on the number of leaders is tight and cannot be improved for arbitrary N and D. In other words, there are graph classes for which we need at least $k = \lceil \frac{N-1}{D} \rceil$ leaders for strong structural controllability, for instance path graphs ($D = N-1$ and $k = 1$), cycle graphs ($D = \lceil N/2 \rceil$ and $k = 2$), complete graphs ($D = 1$ and $k = N-1$).

To construct graphs satisfying the conditions in Theorem 5.1, we again use the notion of PMI sequences of distance-to-leaders vectors along with the result in Theorem 1.1. For any N and D, we construct graphs that give a full PMI sequence wih k leaders, thus, graphs with strong structural controllability. Moreover, we want k to be as small as possible, and note that for certain N and D, k is $\lceil \frac{N-1}{D} \rceil$ as discussed previously. In fact, we first show that if a graph has a full PMI sequence with k leaders, then $k \geq \lceil \frac{N-1}{D} \rceil$.

Theorem 5.2: Let G be a graph with N nodes, diameter D, and k leaders such that G has a full PMI sequence, then $N \leq (kD+1)$.

A proof of the above result is available in [14].

Thus, to have a full PMI sequence, we cannot do better than selecting a minimum of $k = \lceil \frac{N-1}{D} \rceil$ leaders. Next, we show that for any N and D, we can construct graphs that have full PMI sequences (and hence strong structural controllability) with $\lceil \frac{N-1}{D} \rceil$ leaders. Our approach is as follows:

First, for given positive integers k and D, we construct a sequence of $N = kD+1$ vectors satisfying the PMI property. Each vector in the sequence is k-dimensional and contains values from the set $\{0, 1, \cdots, D\}$.

Second, we construct a graph with N nodes and k leaders such that the distance-to-leader vectors of nodes are exactly same as the vectors obtained in the above step. Thus, the constructed graph has a full PMI sequence of distance-to-leader vectors. The maximum distance between any leader and non-leader node in such a graph will be D.

Third, we densify the above graph, that is, maximally add edges to the graph while ensuring that the distance-to-leader vectors of nodes do not change. Consequently, we get graphs with N nodes, D diameter and k leaders. Adding edges always reduces K_f and hence, improves robustness. The graphs obtained have full PMI sequences of distance-to-leader vectors, and are strong structurally controllable.

To construct sequences, we state the following proposition.

Proposition 5.3: Let $S(i,k)$ define the following set of k vectors in $\mathbb{Z}^k$:

$$S(i,k) = \begin{bmatrix} i & i+1 & \dots & i+1 \\ i & i & \dots & i+1 \\ \vdots & \vdots & \ddots & \vdots \\ i & i & \dots & i \end{bmatrix},$$

then the following sequence of $kD+1$ vectors in $\mathbb{Z}^k$ defines a PMI sequence for any positive integers k and D.

$$\begin{bmatrix} \begin{matrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{matrix} & S(1,k) & S(2,k) & \dots & S(D-1,k) & \begin{matrix} D \\ D \\ D \\ \vdots \\ D \end{matrix} \end{bmatrix} \tag{12}$$

Graph Construction: Next, we construct a graph $\mathcal{M}$ with k leaders and $N = kD+1$ nodes whose distance-to-leader vectors are same as in (12). To do so, consider a vertex set

$$V = \{\ell_i\} \cup \{x\} \cup \{u_{i,j}\},$$

where $i \in \{1, 2 \cdots, k\}$ and $j \in \{1, 2, \cdots, D-1\}$. Nodes in $\{\ell_1, \ell_2, \cdots, \ell_k\}$ are leaders. We connect these vertices as follows:

- All leader nodes ℓ_i are pair-wise adjacent and induce a clique.
- x is adjacent to each ℓ_i and $u_{i,1}$, $\forall i \in \{1, \cdots, k\}$.
- For each $i \in \{2, \cdots, k\}$, $u_{i,1}$ is adjacent to leaders ℓ_p, $\forall p \in \{i, i+1, \cdots, k\}$.
- For each $i \in \{1, \cdots, k\}$, $u_{i,j}$ is adjacent to $u_{i,j+1}$, where $j \in \{1, \cdots, D-1\}$.

The above construction is illustrated in Figure 2.

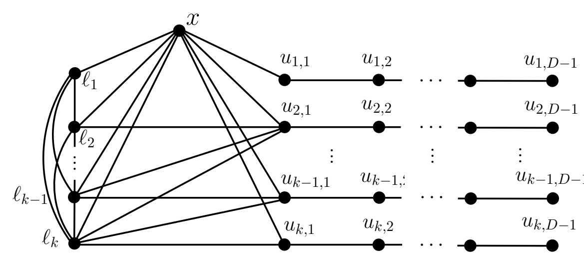

Fig. 2: Graph $\mathcal{M}$ with $N = kD+1$ nodes, where k is the number of leaders and D is the maximum distance between a leader ℓ_i and some other node. Here, $d(\ell_i, u_{1,D-1}) = D, \forall i$.

Next, we compute the distance-to-leader vectors of nodes in $\mathcal{M}$ as follows:

- For all $i \in \{1, \cdots, k\}$, the distance-to-leaders vector of ℓ_i is a vector of all 1's except at the i^{th} index , where it is 0. For the node x, it is a vector of all 1's.
- For node $u_{1,j}$, where $j \in \{1, \cdots, D-1\}$, it is a vector in which all entries are $j+1$.
- For node $u_{i,j}$, where $i \in \{2, \cdots, k\}$ and $j \in \{1, \cdots, D-1\}$, the distance-to-leaders vector has first $(i-1)$ entries equal to $(j+1)$ and the remaining entries are j.

Next, we consider the following sequence of nodes,

$$[\ell_1,\ \ell_2,\ \cdots,\ \ell_k,\ x,\ u_{2,1},\ u_{3,1},\ \cdots,\ u_{k,1},\ u_{1,1},\ u_{2,2},\ u_{3,2},\ \cdots,\ u_{k,2},\ u_{1,2},\ u_{2,3},\ u_{3,3},\ \cdots,\ u_{k,3},\ u_{1,3},\ \cdots,\ u_{2,D-1},\ u_{3,D-1},\ \cdots,\ u_{k,D-1},\ u_{1,D-1}]. \tag{13}$$

If the distance-to-leader vectors of nodes in $\mathcal{M}$ are arranged in the same order as in (13), we get the same sequence as in (12), which is a PMI sequence of length N. Hence, $\mathcal{M}$ has a full PMI sequence, and is strong structurally controllable.

Example: Consider the graph in Figure 3, with $N = 21$ nodes and $k = 4$ leaders. For any leader ℓ_i, the maximum distance between ℓ_i and any other node is $D = 5$. A full PMI sequence of distance-to-leaders vectors is given below. Note that for each vector, there is an index (row index of the circled value) such that the corresponding row value of all the subsequent vectors in the sequence is strictly larger than the circled value, thus constituting a full PMI sequence.

$$\begin{bmatrix} \ell_1 & \cdots & \ell_4 & x & u_{2,1} & u_{3,1} & \cdots & u_{3,4} & u_{4,4} & u_{1,4} \\ \textcircled{0} & & 1 & \textcircled{1} & 2 & 2 & & 5 & 5 & \textcircled{5} \\ 1 & \cdots & 1 & 1 & \textcircled{1} & 2 & \cdots & 5 & 5 & 5 \\ 1 & \cdots & 1 & 1 & 1 & \textcircled{1} & \cdots & \textcircled{4} & 5 & 5 \\ 1 & & \textcircled{0} & 1 & 1 & 1 & & 4 & \textcircled{4} & 5 \end{bmatrix}$$

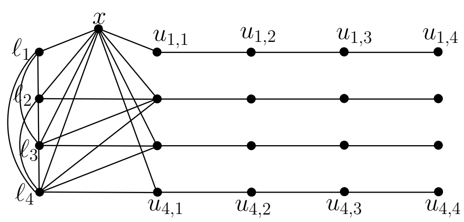

Fig. 3: A graph with 21 nodes and 4 leaders.

Adding Edges to Graph $\mathcal{M}$: We note that removing an edge from $\mathcal{M}$ could change the distance-to-leader vectors of nodes. However, we can add edges to $\mathcal{M}$ to improve its robustness by lowering the Kirchhoff index. Next, we construct a new graph $\bar{\mathcal{M}}$ by maximally adding edges to $\mathcal{M}$ while preserving distances between leaders and all other nodes. Consequently, all distance-to-leader vectors and resulting PMI sequence of $\mathcal{M}$ and $\bar{\mathcal{M}}$ are same. We describe the addition of new edges below.

- For a fixed j, all the nodes in $u_{i,j}$, where $i \in \{i, \cdots, k\}$ induce a clique.
- Each $u_{i,j}$ is adjacent to $u_{1,j-1}$.
- For a fixed $j > 1$, each $u_{i,j}$, where $i > 1$, is adjacent to $u_{p,j-1}, \forall p \in \{i+1, \cdots, k\}$.

An example of $\bar{\mathcal{M}}$ obtained from $\mathcal{M}$ for $N = 21$, $D = 5$, and $k = 4$ is shown in Figure 4.

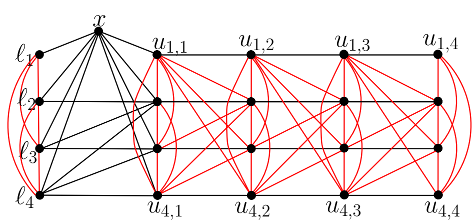

Fig. 4: Construction of $\bar{\mathcal{M}}$ by adding a maximal edge set (red edges) to $\mathcal{M}$. Here $N = 21$, $k = 4$ and $D = 5$.

Proposition 5.4: For fixed k and D, the graph $\bar{\mathcal{M}}$ is maximal in the sense that adding any new edge would change the distance-to-leader vector of some node.

Proof – Available in [14].

Next, we state the following:

Proposition 5.5: If D is the maximum distance between a leader node ℓ_i and some other node in $\mathcal{M}$, then D is the diameter of $\bar{\mathcal{M}}$ constructed from $\mathcal{M}$.

Remark 3 - So far, we have assumed that $N = kD + 1$ for some integer k. However, we can obtain the desired graph for any N by modifying $\bar{\mathcal{M}}$. Let N_a be the actual number of nodes, and D be the desired diameter, then we construct a graph $\bar{\mathcal{M}}$ with $N = kD + 1$ nodes where $k = \lceil \frac{N_a - 1}{D} \rceil$. We need at least that many leaders to have a graph with a full PMI sequence (Theorem 5.2). Since $N_a < N$, we need to delete $(N - N_a)$ nodes from $\bar{\mathcal{M}}$. We delete the required number of nodes in the following order: first, we delete the nodes (in the same order) $u_{1,D-1}, u_{k,D-1}, u_{k-1,D-1}, u_{k-2,D-1}, \cdots, u_{3,D-1}$, then $u_{1,D-2}, u_{k,D-2}, u_{k-1,D-2}, u_{k-2,D-2}, \cdots, u_{3,D-2}$, and so on until the total number of nodes in the remaining graph is N_a. Note that the nodes $u_{2,D-i}$, where $i \in \{1, 2, \cdots\}$ are not deleted to preserve the diameter D. In fact, it is easy to verify that as a result of nodes deletion, the distance-to-leaders vectors of nodes in the remaining graph remain the same as in the original graph, and hence the longest PMI sequence of distance-to-leaders vectors of the nodes in the remaining graph has a length N_a (full PMI sequence). Thus, we can state the following proposition.

Proposition 5.6: For any N and D, there exist graphs that have full PMI sequences with $k = \lceil \frac{N-1}{D} \rceil$ leaders.

Since having full PMI sequences is a sufficient condition for strong structural controllability (Theorem 1.1), and since we can construct graphs with full PMI sequences for any N and D with $k = \lceil \frac{N-1}{D} \rceil$ leaders (Proposition 5.6), we get the result in Theorem 5.1 as a direct consequence.

B. Robustness of Maximally Controllable Networks

Here, we compare the robustness of maximally controllable graphs for a given N and D as obtained above with the the robustness of maximally robust graphs, that is clique chains. Although we know that for given N and D, maximally robust graphs belong to $\mathcal{G}_D(1, n_2, \cdots, n_D, 1)$ where $N = 2 + \sum_{i=2}^{D} n_i$; we don't know the exact values of n_i's in general and need to compute them numerically. In Table I, we choose the same values of N and D as in Table 1 in [3], wherein the K_f of optimal (unweighted) clique chains corresponding to the selected N and D are given. We compare these values with the K_f of the maximally controllable graphs (unweighted) for the same N and D. It is seen that the K_f of maximally controllable graphs is roughly the double of the K_f of the corresponding clique chain, especially for the larger D values.

VI. Conclusions

Networks that exhibit higher robustness to noise and structural changes typically require many leader nodes (inputs) to be completely controllable. For a fixed number of nodes N, complete graphs are maximally robust but require $(N - 1)$ leaders for complete controllability. At the same time, path graphs require only one leader for complete controllability,

TABLE I: K_f of optimal clique chains and maximally controllable graphs $\bar{\mathcal{M}}$.

N	D	k	$K_f(\mathcal{G}^*_D)$ [3]	$K_f(\bar{\mathcal{M}})$
26	2	13	25.08	35.05
	3	9	28.22	49.36
	4	7	37.63	66.08
	5	5	51.90	107.18
	6	5	70.28	109.15
50	2	25	49.04	68.41
	3	17	52.11	95.40
	4	13	64.03	126.22
	5	10	84.31	174.86
	6	9	110.01	202.77
100	2	50	99.02	137.77
	3	33	102.05	193.63
	4	25	117.51	252.58
	5	20	148.11	322.26
	6	17	189.44	393.08
122	2	61	121.01	168.28
	3	41	124.04	231.81
	4	31	140.68	300.42
	5	25	175.11	376.06
	6	21	222.84	460.38

however, such graphs are minimally robust. We observed a similar relationship between controllability and robustness if we also fix the diameter D of a graph along with N vertices. Clique chains are optimal from the robustness perspective for given N and D. However, they require a large number of leaders for strong structural controllability. On the other hand, for arbitrary N and D, we can construct graphs that are strong structurally controllable with at most $\lceil \frac{N-1}{D} \rceil$ leaders, which is a sharp bound. However, such graphs are much less robust compared to optimal clique chains with the same N and D. In the future, we aim to explore graph operations that maximally improve one of the two properties while minimally deteriorating the other one.

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APPENDIX

Pseudo-Monotonically Increasing (PMI) Sequence

Let $\mathcal{S} = \begin{bmatrix} S_1 & S_2 & \cdots & S_N \end{bmatrix}$ be a sequence of vectors where $S_i \in \mathbb{R}^k$, $\forall i$. Moreover, we denote the j^{th} entry of S_i by $S_{i,j}$. $\mathcal{S}$ is a PMI sequence if for each $S_i \in \mathcal{S}$, there exists an index $\alpha(i) \in \{1, 2, \cdots, k\}$ such that

$$S_{i,\alpha(i)} < S_{w,\alpha(i)}, \; \forall w > i.$$

In our context, we are interested in finding a longest PMI sequence of distance-to-leaders vectors of nodes in a leader follower graph as defined below.

Let $G(V, E)$ be a leader follower graph with k leader nodes $\ell_1, \ell_2, \cdots, \ell_k$. For each node $i \in V$, we define a *distance-to-leaders* vector $S_i \in \mathbb{Z}^k_+$ such that the j^{th} entry of S_i is the distance of node i with the leader j, that is,

$$S_i = \begin{bmatrix} d(i, \ell_1) & d(i, \ell_2) & \cdots & d(i, \ell_k) \end{bmatrix}^T.$$

An illustration of the distance-to-leaders vectors is shown in Figure 5. A PMI sequence of distance-to-leaders vectors is,

$$\mathcal{S} = \left[\begin{bmatrix} \textcircled{0} \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ \textcircled{0} \end{bmatrix}, \begin{bmatrix} \textcircled{1} \\ 1 \end{bmatrix}, \begin{bmatrix} \textcircled{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ \textcircled{1} \end{bmatrix} \right].$$

Note that for each vector, there is an index (of the circled value) such that values of all subsequent vectors at the corresponding index are greater than the circled value.

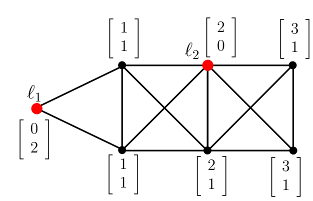

Fig. 5: A graph with two leaders ℓ_1, ℓ_2 and distance-to-leaders vectors of all nodes.

Theorem 1.1: [5] The dimension of SSC is lower bounded by the length of longest PMI sequence of distance-to-leaders vectors.

If the longest PMI sequence of distance-to-leaders vectors in a graph has a length equal to the number of nodes, we say that the graph has a ***full PMI sequence***. Hence, if a network graph has a full PMI sequence, then it is strong structurally controllable.