International Journal of Modern Computation, Information and Communication Technology
2020;3(3-4):30-35.
ISSN: 2581-5954
http://ijmcict.gjpublications.com

Research Article

Fredholmness Conditions for Operators Perturbed by Orthogonal Idempotents in Banach Spaces

K. W. Nyaluke, N. B. Okelo*, O. Ongati

School of Mathematics and Actuarial Science,
Jaramogi Oginga Odinga University of Science and Technology,
P. O. Box 210-40601, Bondo-Kenya.

*Corresponding author's e-mail: bnyaare@yahoo.com

Abstract

Let H be an infinite dimensional complex Hilbert space and $F_{01}(H)$ the set of all Fredholm operators perturbed by orthogonal idempotents in Banach space. In the present paper, we determine the conditions under which Fredholm operators retain Fredholmness when perturbed by orthogonal idempotents in Banach spaces.

Keywords: Fredholm operators; Hilbert space; Orthogonal idempotents; Banach spaces.

Introduction

The study of Fredholm operators began decades back when mathematicians considered integral equations on Hilbert spaces [1-3]. Since then Fredholm operators became the most important classes of linear operators in mathematics and the different classes of operators like semi-groups, class of Fredholm operators such as classes of upper and lower semi-Fredholm operators, and ideals of operators such as classes of finite-dimensional and compact operators have been studied by several mathematicians and general characterizations of each group obtained. Later on, the perturbation theory of linear operators was developed by several researchers like Rayleigh and Schrondinger among others and the perturbations of Fredholm operators was done and nice results obtained.

Recently, [4] worked on the perturbation classes of semi-Fredholm and Fredholm operators, also [5] characterized some results on Fredholm operators, essential spectra, and application while [6] provided a notion concerned with perturbation of the group generalized inverse for the class of bounded operators in Banach spaces and determined the index of $B-$Fredholm operators and generalization of Weyl theorem. Later on, it was noted in [7] noted on preservation of generalized Fredholm Spectra in Berkanis sense. Similarly, perturbations of Fredholm operators have been thoroughly researched in [8] and [9].

Furthermore, in [10] the authors worked on perturbation classes for semi-Fredholm operators on $L_p(\mu)-$spaces and on the other hand [11] surveyed on the perturbation classes problem for semi-Fredholm and Fredholm operators, functional analysis, approximation and Computation and further researched extensively on perturbations of Fredholm operators in [3]. In [7] the authors studied finite-dimensional Perturbations of Linear operators in Banach spaces and some applications to boundary integral equations, engineering analysis with boundary elements. Several properties of Fredholm composition operators have been discussed in detail in [1] among many other researchers. Many of those researchers characterized Fredholm operators and determined properties for the invertibility of compact operators in Banach spaces but not for the Fredholm operators perturbed by orthogonal idempotent in Banach spaces. In the present paper we determine conditions under which Fredholm operators retain Fredholmness when perturbed by orthogonal idempotents in Banach spaces for the class of $F_{OI}(H)$. We have reviewed some basic concepts which are useful to our work as they are outlined in Research methodology.

Received: 21.02.2020; Received after Revision: 10.03.2020; Accepted: 10.03.2020; Published: 16.04.2020

Research methodology

Definition 2.1

Let X and Y be Banach spaces, a linear operator $T : X \to Y$ is a Fredholm operator if and only if:

i. T is closed.
ii. The domain of T is dense in X.
iii. $\alpha(T)$, the dimension of the null space $N(T)$ of T is finite.
iv. The range of T is closed in Y.
v. $\beta(T)$, the co-dimension of $R(T) \in Y$ is finite.

Definition 2.2

If $P \in B(H)$ satisfies $P^2 = P$ then $\ker(P) = Ran(P)^{\perp}$ then P is called a projection. If $P(H)$ is a closed subspace of H then, for each closed subspaces $\mu \in H$ there exists an orthogonal projection P such that $PH = \mu$. For any family $g_j, h_{j=1}^n \subset H$ and $f \in H$. Let $A_n f = \sum \langle f, g \rangle h_j$. Then $A_n \in B(H)$, and $Ran(A_n) < \infty$ are finite rank operators.

Definition 2.3

Given two Banach spaces X and Y, the set of all upper semi-Fredholm operators is defined by $\Phi_+(X,Y) := \{T \in L(X,Y) : \alpha(T) < \infty\}$ and $T(X)$ is closed, while the set of all lower semi-Fredholm operators is defined by $\Phi_-(X,Y) := \{T \in L(X,Y) : \beta(T) < \infty.\}$ The set of all semi-Fredholm operators is defined by $\Phi_\pm(X,Y) = \Phi_+(X,Y) \cup \Phi_-(X,Y)$ and the class of Fredholmm operators id defined by $\Phi_\pm(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$.

Definition 2.4

Let $T : H \to H$ be linear operator. Then T is said to be normal if $T^*T=TT^*$ and self-adjoint operator if $T=T^*$

Definition 2.5

If $P \in B(H)$ satisfies $P^2 = P$ then P is said to be idempotent.

Results and discussions

Under this section we determine conditions for the Fredholmness in $F_{OI}(H)$.

Proposition 3.1.

Let $T \in F_{OI}(H)$. Fredholmness is retained by T if the identity map $(F_{OI}(H), \mu) \alpha\ (F_{OI}(H), r)$ is continuous and the following conditions are equivalent.

i. $r(T_n, T) \to 0$.
ii. $\|g(T_n) - g(T)\| \to 0, \quad \forall g \in [F_{OI}(H)]_0$.

Proof.

Let $q_\pm \in [F_{OI}(H)]_0$, then it shows that (ii) implies (i) such that $r(T_n, T) = \|g_-(T_n) - g_-(T)\| + \|g_+(T_n) - g_+(T)\|$ and if $r(T_n, T) \to 0, \quad \forall q \in Q$. Let $q \in [F_{OI}(H)]_0$ and Q dense in $[F_{OI}(H)]_0$, $\forall \varepsilon > 0$ we have $q \in Q$ such that $\|g - q\| \le \frac{\varepsilon}{3}$ and $n(\varepsilon) > 0 \quad \forall n \ge n(\varepsilon)$ gives $\|q(T_n) - q(T)\| \le \frac{\varepsilon}{3}$.

Therefore,

$$\|g(T_n) - g(T)\| \le \|g(T_n) - q(T_n)\| + \|q(T_n) - q(T)\| + \|q(T) - g(T)\| \le \varepsilon.$$

Proposition 3.2

Let $T \in F_{OI}(H)$. Fredholmness is retained by T if $\gamma \in F_{OI}(H)$ such that $\gamma(w) \equiv 1$ for all $w >> 1$, while $\gamma(w) = 0$ if whenever $w << 1$ and the following are equivalent.

i. $r(T_n, T) \to 0$.
ii. $\|g(T_n) - g(T)\| \to 0, \quad \forall g \in [F_{OI}(H)]_0$.
iii. $r(T_n, T) \to 0$ and $\|\gamma(T_n) - \gamma(T)\| \to 0$.

Proof.

Let $r \in F_{OI}(H)$ and r represented as $r(w) := \frac{w}{(1+w^2)^{\frac{1}{2}}}$. Since the sub algebra spanned by $[F_{OI}(H)]_0$ and r dense in $F_{OI}(H)$. Therefore, the equivalence of (i) implies (iii) is proved in Proposition 3.1.

Theorem 3.3.

Let $T \in F_{OI}(H)$ and $Ran(T) \cap \Re \neq 0$. Then suppose T_n is a sequence of Fredholm operators satisfying the following conditions:

i. $D(T) \subset D(T_n)$.

ii. There exists a sequence Fredholm operators of positive numbers $\lambda_n \to 0$ such that $\|(T_n a)\| \leq \lambda_n(\|Ta\| + \|a\|)$; $\forall a \in D(T)$.

Then $T + T_n \in F_{OI}(H)$ $\forall n >> 0$ and $r(T + T_n, T) \to 0$.

Proof.

Let $T_n := (T + T_n)$ and $r(T + T_n) \to 0$, we have $T + T_n \in F_{OI}(H)$ $\forall n > 0$. Therefore, if we let $\alpha \in Ran(T) \cap \Re$ and $I = [\alpha - \varepsilon, \alpha + \varepsilon]$ such that $I \subset Ran(T)$. Then, if n is large, we can have $I \subset Ran(T_n)$ $\forall n >> 0$ such that $\gamma \in F_{OI}(H)$. If $\gamma(w) \equiv 1$, $\forall w \geq \alpha + \varepsilon$ and $\gamma(w) \equiv 0$, $\forall w \leq \alpha - \varepsilon$. Hence $\|\gamma T_n - g(T)\| \to 0$. Therefore, we invoke the proof of Proposition 3.2 to show that $r(T_n, T) \to 0$.

Theorem 3.4

Let $J_1, J_2 \in F_{OI}(H)$ and $T \in F_{OI}(H)$ be defined by $T := \alpha J_1 + \beta J_2 - \lambda J_1 J_2$. Then T retains Fredholmness if $\alpha\beta \neq 0$, $ind(T) = ind(J_1 + J_2)$. Moreover, Fredholmness of T is independent of the choice of α, β and λ.

Proof.

Let J_1 and J_2 be two orthogonal idempotents. Then T is Fredholm if and only if $\alpha T^{-1} J_1 T + \beta T^{-1} J_2 T - \lambda T^{-1} J_1 T(T^{-1} J_1 T)$ is Fredholm. Therefore, without the loss of generality of Fredholmness of T, we can assume that J_1 and J_2 are orthogonal idempotents. For example, let J_2 be a positive orthogonal idempotent, we can have $J_1 = \begin{pmatrix} I & J_{11} \\ 0 & 0 \end{pmatrix}$ and $J_2 = \begin{pmatrix} J_{22} & J_{22}^{\frac{1}{2}} \\ J_{33}^{\frac{1}{2}} D^* J_{22}^{\frac{1}{2}} & J_{33} \end{pmatrix}$. Then, decomposition of $F_{OI}(H) = Ran(J_1) \oplus Ran(J_1)^{\perp}$, where J_{22} and J_{33} are positive operators on $Ran(J_1)$ and $Ran(J_1)^{\perp}$ respectively, while D is contraction operator from $Ran(J_1)^{\perp} \to Ran(J_1)$. The matrix J_{22} and J_{33} is computed as follows

$$J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & J_{22} \end{pmatrix}, \quad J_{33} = \begin{pmatrix} J_{33} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to Fredholm decomposition of $Ran(J_1) = N(J_{22}) \oplus N(I - J_{22}) \oplus (Ran J_1) - (N(J_{22}) \oplus N(I - J_{22}))$ and $Ran(J_1)^{\perp} = Ran(J_1)^{\perp} - N(I - J_{33}) \oplus N(I - J_{33}) \oplus J_{33}$. Then if we let $[F_{OI}(H)]_0 = N(J_{22})$, $[F_{OI}(H)]_1 = N(I - J_{22})$, $[F_{OI}(H)]_2 = Ran(J_1) - N(J_{22}) \oplus N(I - J_{22})$, $[F_{OI}(H)]_3 = Ran(J_1)^{\perp} - N(J_{33}) \oplus N(I - J_{22})$, $[F_{OI}(H)]_4 = N(I - J_{33})$, and $[F_{OI}(H)]_5 = N(J_{33})$. Hence, the J_1 and J_2 have the matrix representation as:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & J_{11} & J_{22}^{\frac{1}{2}} D_1 J_{22}^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & J_{22}^{\frac{1}{2}} D^* J_{11}^{\frac{1}{2}} & J_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$J_2 = \begin{pmatrix} I & 0 & 0 & J_{11} & J_{12} & J_{13} \\ 0 & I & 0 & J_{21} & J_{22} & J_{23} \\ 0 & 0 & I & J_{31} & J_{32} & J_{33} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, if for some contraction D_1 from $[F_{OI}(H)]_3$ to $[F_{OI}(H)]_2$ with respect to Fredholm decomposition of $F_{OI}(H) = \oplus_{i=0}^{5} [F_{OI}(H)]_i$

such that $J_0 = \begin{pmatrix} J_{11} & J_{11}^{\frac{1}{2}} \\ J_{22}^{\frac{1}{2}} D_1^* J_{11}^{\frac{1}{2}} & J_{22} \end{pmatrix}, \quad \forall$ $J_1 \in I_O(H)$ and $J_0 \in [F_{OI}(H)]_2 \oplus [F_{OI}(H)]_3 \quad \forall \quad J_0 = J_0^2.$ Similarly, this

$$J_{11} = J_{11}^2 + J_{11}^{\frac{1}{2}} D_1 J_{22} D_1^* J_{11}^{\frac{1}{2}},$$

$$J_{11}^{\frac{1}{2}} D_1 J_{22}^{\frac{1}{2}} = J_{11}^{\frac{1}{2}} D_1 J_{22}^{\frac{1}{2}} + J_{11}^{\frac{1}{2}} D_1 J_{22}^{\frac{3}{2}},$$

$$J_{22}^{\frac{1}{2}} D_1^* J_{11}^{\frac{1}{2}} = J_{22}^{\frac{3}{2}} D_1^* J_{11}^{\frac{1}{2}} + J_{22}^{\frac{1}{2}} D_1^* J_{11}^{\frac{3}{2}}$$

and $J_{22} = J_{22} + J_{22}^{\frac{1}{2}} D_1^* J_{11}^{\frac{1}{2}} D_1 J_{22}^{\frac{3}{2}}$ has been obtained by use of the injectivity of J_{11}, $I - J_{11}$, J_{22} and $I - J_{22}$. Therefore, $D_1 D_1^* = I, \quad D_1^* D_1 = I$ and $J_{22} = D_1^*(I - J_{11})D_1$. Then,

$$T : \alpha J_1 + \beta J_2 - \lambda J_1 J_2 = \begin{pmatrix} V_{11} & 0 & V_{13} & V_{14} & V_{15} & V_{16} \\ 0 & V_{22} & V_{23} & V_{24} & V_{25} & V_{26} \\ 0 & 0 & V_{11} & V_{12} & V_{35} & V_{36} \\ 0 & 0 & V_{21} & V_{22} & 0 & 0 \\ 0 & 0 & 0 & V_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $V_{11} = \alpha I,$

$$V_{13} = \lambda J_{11} J_{22}^{\frac{1}{2}} D_1^* J_{11}^{\frac{1}{2}},$$

$$V_{14} = \alpha J_{11} - \lambda J_{11} J_{22}, \quad V_{15} = \alpha J_{12} - \lambda J_{12},$$

$$V_{16} = \alpha J_{13}, \quad V_{22} = (\alpha + \beta - \lambda)I,$$

$$V_{23} = -\lambda J_{21} J_{22}^{\frac{1}{2}} D_1^* J_{11}^{\frac{1}{2}},$$

$$V_{24} = \alpha J_{21} - \lambda J_{21} J_{22},$$

$$V_{25} = \alpha J_{21} - \lambda J_{22}, \quad V_{35} = \alpha J_{32} - \lambda J_{32},$$

$$V_{26} = \alpha J_{23}, \quad V_{36} = \alpha J_{33}, \quad V_{55} = \beta I.$$

Therefore, $\alpha J_1 + \beta J_2 - \lambda J_1 J_2$ is Fredholm if and only if $I - J_{11} \in F_{OI}(H)$ and $I - J_{31} D_1^* (I - J_{11})^{\frac{1}{2}}$ is Fredholm. Since $T \in F_{OI}(H)$ and $M = (\alpha J_{11} + \beta J_2 - \lambda J_1 J_2)T - I$ to be a Fredholm operator, we have $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ and $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$ on

$$F_{OI}(H) = Ran(J_1) \oplus Ran(J_1)^{\perp},$$

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} I + M_1 & M_2 \\ M_3 & I + M_4 \end{pmatrix}$$

that gives either $\beta J_{22}^{\frac{1}{2}} D_1^* J_{21}^{\frac{1}{2}} T_2 + \beta J_{22} T_4 = I + M_4$. This shows that $J_{22}^{\frac{1}{2}}$ is Fredholm. The Fredholmness of $\alpha J_1 + \beta J_2 - \lambda J_1 J_2$ is equivalent to that of $\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$ which is also equivalent to

$$U_{11} - U_{22}^* U_{21} = \alpha I + \beta J_{11} - \left(\alpha J_{31} \beta J_{11}^{\frac{1}{2}} \left(D_1 J_{22}^{\frac{1}{2}} \right) \left(\beta J_{22}^{\frac{1}{2}} \right) \beta J_{22}^{\frac{1}{2}} \left(D_1^* J_{11}^{\frac{1}{2}} \right) \right)$$

$$= \alpha I + J_{11} - \left(\alpha J_{31} + \beta J_{11}^{\frac{1}{2}} D_1 D_1^* (I - J_{11})^{\frac{1}{2}} D_1 D_1^* \left((I - J_{11})^{\frac{-1}{2}} D_1 D_1^* J_{11} \right)^{\frac{1}{2}} \right).$$

Then equal to $\alpha(I - J_{31}) D_1^* (I - J_{11})^{\frac{1}{2}} J_{11}^{\frac{1}{2}}$. Hence the equivalence of Fredholmness of $\alpha J_1 + \beta J_2 - \lambda J_1 J_2$ and $J_1, J_2 \in F_{OI}(H)$, therefore,

$$ind(\alpha J_1 + \beta J_2 - \lambda J_1 J_2) = ind\left(\alpha(I - J_{31}) D_1^* (I - J_{11})^{\frac{1}{2}} J_{11}^{\frac{1}{2}} \right) = ind(J_1 + J_2).$$

Corollary 3.5

Let $J_1, J_2 \in F_{OI}(H)$ such that J_1 and J_2 are orthogonal idempotents. Then the following hold:

i. the invertibility of $T \in F_{OI}(H)$ is independent of the choice of α, β and λ $\forall \alpha\beta \neq 0$.

ii. the invertibility of $T \in F_{OI}(H)$ is equivalent to the invertibility of $\alpha J_1, \beta J_2$ and $\alpha, \beta, \lambda \in C$ $\alpha\beta \neq 0$.

Proof.

Let $\alpha_0 J_1 + \beta_0 J_2 - \lambda_0 J_2 J_1$ to be invertible Fredholm operators and for some $\alpha_0, \beta_0, \lambda_0 \in C \quad \forall \quad \alpha_0 \beta_0 \neq 0$. Therefore, $\alpha_0 J_1 + \beta_0 J_2 - \lambda_0 J_2 J_1$ is Fredholm with the nullity, and its limit equal to zero and from

Theorem 3.4 it follows that $T \in F_{OI}(H)$ is invertible for all $\alpha, \beta, \lambda \in C$ with $\alpha\beta \neq 0$ and $\lambda_0 = 0$.

Lemma 3.6.

Let $T \in F_{OI}(H)$, then $\ker(T^*) = Ran(T)^{\perp}$ since $Dom(T)$ is dense and retains Fredholmness defined on $(T, Dom(T))$.

Proof.

Let $g \in \ker(T^*)$ and $T^* g = 0 \quad \forall g \in Dom(T)$. Then if for $g \in Dom(T)$ we show that $0 = \langle T^* g, t \rangle = \langle g, Tt \rangle$ then it follows that $g \in Ran(T)^{\perp}$. Conversely, if $g \in Ran(T)^{\perp}$, then for all $g \in Dom(T)$ we have $\langle g, Tt \rangle = 0 = \langle 0, t \rangle$ such that $g \in Dom(T^*)$ and $T^* g = 0$.

Theorem 3.7.

Let $T \in F_{OI}(H)$ be compact operators defined by $T : R \to A$ that retains Fredholmness, then the following conditions are satisfied.

i. T is Fredholm
ii. $T \in F_{OI}(H)$ and $L \in B(A, R)$ there exists compact operators K_1 and K_2 such that $LT = I + K_1$, $TL = I + K_2$.

Proof.

Let $L, K_1, K_2 \in F_{OI}(H)$ be the identity compact Fredholm operator, then $\ker(T)$ and the $co\ker(T)$ are finite dimensional. Therefore, T retains its Fredholmness. Similarly, if $T \in F_{OI}(H)$, we can have the complement of $R_1 \ker(T)$ and $A_1 \in \operatorname{Im}(T)$ such that $A_1 = A|_{R_1}$ is an isomorphism from R_1 into $\operatorname{Im}(T)$. Also if $L = (A_1)^{-1}$ on $\operatorname{Im}(T)$ and $L = 0$. Therefore, the resulting K_1 is an orthogonal idempotent onto $\ker(T)$ and $I + K_2$ is also an orthogonal idempotent onto $\operatorname{Im}(T)$. Hence $LT = I + K_1$ and $TL = I + K_2$.

Conclusions

In summary, we have determined conditions under which Fredholm operators retains Fredhomness when perturbed by orthogonal idempotents in Banach spaces for the class of $F_{OI}(H)$. We have shown that under certain conditions Fredholmness is retained by T if the identity map $(F_{OI}(H), \mu) \alpha \ (F_{OI}(H), r)$ is continuous.

Conflicts of interest

Authors declare no conflict of interest.

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