Ontario Math Circles
Third Annual ARML Team Selection Test
Solutions

1. Evaluate the sum $0+1+2-3-4-5+6+7+8-\cdots-2019$.
In every block of 6 the total is -9. Thus, the answer is $336\times(-9)+2016+2017+2018-2019=1008$.

2. What is the smallest prime divisor of $2017^{2017}+2019^{2019}$?
The sum of the two numbers is even. Therefore, the answer is 2.

3. Determine the number of ways to assign 3 red and 3 blue to the integers $1,2,3,4,5,6$ such that no two consecutive integers have the same colour.
This is only possible if $1,3,5$ are one color and $2,4,6$ are another. There are 2 ways to do this.

4. Let l be the line that passes through the origin and $(1,2)$. Find the shortest distance from the point $(0,10)$ to l.
The equation of line l is $y=2x$. The perpendicular line to l through $(0,10)$ is $y=-\frac{1}{2}x+10$. The intersection of these two lines is $(4,8)$. The distance from $(4,8)$ to $(0,10)$ is $2\sqrt{5}$.

5. Define the operation $*$ with $a*b=a^2-2^b$. Solve the following inequality for all real solutions

$$t*t<t*1\leq t$$

For the first inequality,

$$\begin{aligned} t*t&<t*1\\ t^2-2^t&<t^2-2\\ 2&<2^t\\ 1&<t \end{aligned}$$

For the second inequality,

$$\begin{aligned} t*1&\leq t\\ t^2-2&\leq t\\ t^2-t-2&\leq 0\\ (t-2)(t+1)&\leq 0\\ -1\leq t&\leq 2 \end{aligned}$$

Combining these two inequalities yields $1<t\leq 2$.

6. Find the unit's digit of 3^{3^3}.
The pattern for power of 3 is $3,9,7,1$. Therefore, the unit's digit of $3^{3^3}=3^{27}$ is 7.

7. How many positive integers less than 2018 is divisible by 2 or 3?
It suffice to evaluate

$$\left\lfloor\frac{2017}{2}\right\rfloor+\left\lfloor\frac{2017}{3}\right\rfloor-\left\lfloor\frac{2017}{6}\right\rfloor=1344$$

8. The system of equations

$$\begin{cases} x^2+y^2=100\\ x^2+y^2+180=20(x+y) \end{cases}$$

intersect at two distinct points A and B. Determine the equation of the line that passes through A and B.
Substituting the first equation into the second yields $x+y=14$, which is the equation of the line.

9. Let $f(x) = ax^2 + bx + c$ with $a \neq 0$. If $f(f(0)) = f(0)$, find the smallest possible value of the discriminant of $f(x)$.
First,

$$\begin{aligned} f(f(0)) &= f(0) \\ f(c) &= c \\ ac^2 + bc + c &= c \\ ac^2 + bc &= 0 \\ c(ac+b) &= 0 \end{aligned}$$

Therefore, $c = 0$ or $ac + b = 0$. The discriminant of $f(x)$ is $b^2 - 4ac$. Using the result of part (c), if $c = 0$ then

$$b^2 - 4ac = b^2 \geq 0$$

If $ac + b = 0$ then

$$b^2 - 4ac = b^2 - 4(-b) = b^2 + 4b = b(b+4) \geq -4$$

Therefore, the smallest possible value of the discriminant of $f(x)$ is -4.

10. Consider the list of 2019 numbers $111, 1011, 10011, 100011, 1000011, \ldots$. How many of these numbers are divisible by 33?
Note that none of the numbers are divisible by 11, which implies that the answer is 0.

11. Determine the number of ways to assign red, blue, and green to the integers $1, 2, 3, 4, 5, 6$ such that no two consecutive integers have the same colour.
There are 3 choices for 1 and for each after there are 2 choices. Therefore, the answer is $3 \times 2^5 = 96$.

12. Find all ordered pairs of real numbers (x, y) which satisfy

$$2\log x = \log x^2 + 2\log y$$

$$\begin{aligned} 2\log x &= \log x^2 + 2\log y \\ \log x^2 &= \log x^2 y^2 \\ x^2 &= x^2 y^2 \\ 0 &= x^2(y+1)(y-1) \end{aligned}$$

Since $x, y > 0$ then the only solutions are when $y = 1$ and $x > 0$.

13. Consider the following system of inequalities

$$\begin{cases} -5 < y \leq \lfloor x \rfloor \\ 0 \leq x < 10 \end{cases}$$

Find the area of the region.
Let n be a non-negative integer then for $n \leq x < n + 1$, the area of this region with $-5 < y \leq \lfloor x \rfloor$ is $5 + n$. Therefore, the answer is

$$\sum_{n=0}^{9}(5+n) = \sum_{n=0}^{9} 5 + \sum_{n=0}^{9} n = 50 + 45 = 95$$

14. Find the minimum value of $x6^{\frac{1}{x}} + \frac{1}{x}6^x$ for $x > 0$.
By AM-GM,

$$x6^{\frac{1}{x}} + \frac{1}{x}6^x \geq 2\sqrt{6^{\frac{1}{x}}6^x} = 2\sqrt{6^{x+\frac{1}{x}}} \geq 2\sqrt{6^2} = 12$$

The minimum is achievable when $x = 1$.

15. Find all positive integers n such that $7^n + 147$ is a perfect square.
Clearly fails for $n = 1$ so for $n > 1$ note that $7^n + 147 = 49\left(7^{n-2} + 3\right)$. Here, we want $7^{n-2} + 3$ to be a perfect square, which works when $n = 2$. Otherwise, since $7^k + 3 \equiv 3 \bmod 7$ for $k \in \mathbb{N}$, it follows this cannot be a perfect square (which must be 0,1,2,4 mod 7). Hence, $n = 2$ is the only such integer.

16. In $\triangle ABC$, $\angle B = 90°$, $\angle A = 60°$, and $AB = 1$. Let $\angle B$ be trisected to create three smaller triangles. If r_1, r_2, r_3 are the inradii of these three triangles, find $r_1 + r_2 + r_3$.
Let the trisectors of $\angle B$ be S and T in the order of A, S, T, C. The measurements of length are as follow: $AS = \frac{1}{2}, BS = \frac{\sqrt{3}}{2}, BT = TC = 1, ST = \frac{1}{2}$. The measurements of area are as follows: $[ABS] = [BST] = \frac{\sqrt{3}}{8}$ and $[BTC] = \frac{\sqrt{3}}{4}$. Since $A = sr$ then

$$r_1 + r_2 + r_3 = \frac{\frac{\sqrt{3}}{8}}{\frac{3}{4} + \frac{\sqrt{3}}{4}} + \frac{\frac{\sqrt{3}}{8}}{\frac{3}{4} + \frac{\sqrt{3}}{4}} + \frac{\frac{\sqrt{3}}{4}}{1 + \frac{\sqrt{3}}{2}} = \frac{3\sqrt{3}}{2} - 2$$

17. Compute $\sum_{k=1}^{\infty} \frac{k\pi}{\pi^k}$.

$$\sum_{k=1}^{\infty} \frac{k\pi}{\pi^k} = \pi \sum_{k=1}^{\infty} \frac{k}{\pi^k} = \pi \frac{1/\pi}{(1 - 1/\pi)^2} = \frac{\pi^2}{\pi^2 - 2\pi + 1}$$

18. How many ways are there to choose 3 numbers from $1, 2, 3, \ldots, 10$ such that no 2 of the 3 numbers are consecutive?
Let $f(n)$ be the number of ways to solve the problem with $1, 2, \ldots, n$. Observe that $f(5) = 1$ and $f(n) = f(n-1) + \binom{n-2}{2} - (n-3)$. Following this pattern yields $f(10) = 56$.

19. Find the sum of all positive integers n for which $f(n) = n^4 - 360n^2 + 400$ is a prime number.
First,

$$f(x) = n^4 - 360n^2 + 400 = (n^2 - 20n + 20)(n^2 + 20n + 20)$$

For $f(x)$ to not be prime, the first term must be 1, which occurs when $x = 1$ or $x = 19$. Therefore, $1 + 19 = 20$.

20. If $x > y > 0$ and $2\log_{10}(x - y) = \log_{10} x + \log_{10} y$, what is the value of $\frac{x}{y}$.

$$\begin{aligned}
2\log_{10}(x-y) &= \log_{10} x + \log_{10} y \\
(x-y)^2 &= xy \\
x^2 - 2xy + y^2 &= xy \\
x^2 - 3xy + y^2 &= 0 \\
\left(\frac{x}{y}\right)^2 - 3\left(\frac{x}{y}\right) + 1 &= 0 \\
\frac{x}{y} &= \frac{3 \pm \sqrt{9 - 4(1)}}{2} = \frac{3 \pm \sqrt{5}}{2}
\end{aligned}$$

Since $x > y > 0$, $\frac{x}{y} = \frac{3+\sqrt{5}}{2}$.

21. Find the tens digit of the sum

$$1! + 2! + 3! + \cdots + 2019!$$

The tens digit of $n!$ is 0 for $n \geq 10$. Thus, it suffice to compute the tens digit of

$$1! + 2! + 3! + \cdots + 9!$$

which is 1.

22. In $\triangle ABC$, D is a point on BC such that $\angle BAD = 30°$ and $\angle DAC = 15°$. If $AB = 3\sqrt{2}$ and $AC = 6$, find the length of AD.
Since $[ABD] + [ADC] = [ABC]$ then

$$\begin{aligned}
3\sqrt{2}AD \sin 30° + 6AD \sin 15° &= 18\sqrt{2} \sin 45° \\
3\sqrt{2}AD\frac{1}{2} + 6AD\frac{\sqrt{6} - \sqrt{2}}{4} &= 18\sqrt{2}\frac{1}{\sqrt{2}} \\
AD &= 2\sqrt{6}
\end{aligned}$$

23. Find positive integers a, b if for every $x, y \in [a, b]$ then $\frac{1}{x} + \frac{1}{y} \in [a, b]$.
We need $a \leq b$ and it's necessary that $\frac{1}{x} + \frac{1}{y} \geq a$,then let $x = y = a$,we have $a^2 \leq 2$ i.e.$a = 1$. Let $x = y = b$,we have $b \leq 2$ and we get $b = 1$ or $b = 2$. When $b = 1$, we take $x = y = 1$, $\frac{1}{x} + \frac{1}{y} = 2 > 1 = b$, contradiction. When $b = 2$, $x, y \in [1, 2]$ then $\frac{1}{x} + \frac{1}{y} \geq \frac{1}{2} + \frac{1}{2} = 1$ and $\frac{1}{x} + \frac{1}{y} \leq 2$ works. Thus, $(a, b) = (1, 2)$.

24. Let $a_n = n^2 + 2n + 50$, $n = 1, 2, \ldots$. Let d_n be the largest positive integer that is a divisor of both a_n and a_{n+1}. Find the maximum possible value of d_n for $n = 1, 2, \ldots$.
First, d_n must also divide $a_{n+1} - a_n = 2n+3$, and since $4a_n = 4n^2+8n+200 = (2n+3)^2 - 2(2n+3) + 197$, d_n must divide 197. Hence, the largest value of d_n is 197, which is obtained for $n = 97$.

25. Tom writes down the integers from 1 to 100, inclusive, on a chalkboard and then erases every number that contains a prime digit. Compute the sum of the digits that remain on the chalkboard.
This was ARML 2018 Tiebreaker P3. You went back to study the problems right? Answer was 337.

26. Let the tangent line passing through a point A outside the circle with center O touches the circle at B and C. Let BD be the diameter of the circle. Let the lines CD and AB meet at E. If the lines AD and OE meet at F, find $|AF|/|FD|$.
Let FK be the perpendicular from F to EB. Because $\angle BDC = \angle BOC/2 = \angle BOA$, we conclude that the right triangles $\triangle ABO, \triangle EBD$ are similar and that $AO \parallel ED$. Thus, $EA = AB$. Therefore, in $\triangle EBD$, EO and DA are medians and thus $EF/FO = 2 = EK/KB$. (1) If we set $AK = y$ and $AB = x$, we find from (1) that $(x+y)/(x-y) = 2$ and thus $y = x/3$. Therefore, $AF/FD = AK/KB = y/(x-y) = 1/2$.

27. Let $A = \{z : z^{\binom{5}{2}} = 1\}$ and $B = \{w : w^{\binom{10}{2}} = 1\}$. Determine the number of distinct elements in $\{zw : z \in A, w \in B\}$.
It suffices to compute $\operatorname{lcm}\left(\binom{5}{2}, \binom{10}{2}\right) = 90$.

28. Determine the number of ways to assign each of the integers 1 to 8 to $a_1, a_2, \ldots, a_8$ such that $|a_n - n| \leq 1$ for $n = 1, 2, \ldots, 8$.
Let F_n be the number of ways to assign the integers 1 to n to $a_1, a_2, \ldots, a_n$ such that $|a_n - n| \leq 1$. Note that $F_1 = 1$ and $F_2 = 2$ ($1, 2$ and $2, 1$ are the only two possible assignments). Observe that there are two possible values for a_n, n and $n-1$. If $a_n = n$ then it remains to determine the number of ways to assign the integers 1 to $n-1$ to $a_1, a_2, \ldots, a_{n-1}$ such that $|a_n - n| \leq 1$, which is F_{n-1}. If $a_n = n-1$ then $a_{n-1} = n$ because every number must be to assigned to one of the terms in the sequence and a_{n-1} is the only remaining term to assign n. It remains to determine the number of ways to assign 1 to $n-2$ to $a_1, a_2, \ldots, a_{n-2}$ such that $|a_n - n| \leq 1$, which is F_{n-2}. Therefore, F_n is a recursive relation $F_n = F_{n-1} + F_{n-2}$ and $F_1 = 1$ and $F_2 = 2$. Therefore, there are $F_8 = 34$ possible ways.

29. Find the number of positive integer pairs (m, n) with $m, n \leq 50$ such that $m + n + 1$ is prime and divides $2\left(m^2 + n^2\right) - 1$.
Note $2m^2 + 2n^2 - 1 = 2(m+n)(m+n+1) - (2m+1)(2n+1)$ so $m+n+1$ divides $2m+1$ or $2n+1$ since its prime. WLOG assume $m+n+1$ divides $2m+1$ so there exists a positive integer k such that $k(m+n+1) = 2m+1$. This condition rearranges to $kn + (k-1) = (2-k)m$. Note the left side is positive so the right side must be too, implying $2 - k > 0$ so $k = 1$. Hence, $m + n + 1 = 2m + 1$ so $m = n$. Therefore we wish to find the pairs (m, n) with $1 \leq m, n \leq 100$ such that $2m + 1$ is prime. Since there are 25 odd primes at most 101, it follows there are 25 such pairs.

30. Let AB and CD be two segments length 1. If they intersect at O such that $\angle AOC = 60°$, find the minimum of $AC + BD$.
Connect AC, BD and introduce $CB_1 \parallel AB$, where $CB_1 = AB$. Then ABB_1C is a parallelogram, so $BB_1 = AC$. Connect B_1D then $\triangle CDB_1$ is equilateral. Applying the triangle inequality,

$$AC + BD = BB_1 + BD > DB_1 = CD = 1$$

Note that equality occurs when $A = C$.

31. How many 6-digit numbers whose leftmost digit is 1 have exactly two pairs of identical digits (and no digit occurs three or more times)?
There are $\binom{9}{2}\binom{5}{2}\binom{3}{2}7$ ways in which the 1 is not paired. There are $5 \times 9 \times \binom{4}{2} \times 8 \times 7$ ways in which 1 is one of the paired digits. The total is 22680.

32. Given that G is the centroid of $\triangle ABC$, $GA = 2\sqrt{3}$, $GB = 2\sqrt{2}$, $GC = 2$. Find the area of $\triangle ABC$.
First, extend AG to G' such that $GG' = AG$. Second, show that $BGCG'$ is a parallelogram. Third, show that $\triangle GBP$ is right and compute the area of $[GBC]$. Therefore, $[ABC] = 3[GBC] = 6\sqrt{2}$.

33. Find the number of positive integers n such that $n^2 \leq 2019$ is the product of all positive proper divisors of n.
Note $n = 1$ works. Otherwise, we have $\prod_{d|n} d = n^{\tau(n)/2}$ so $\tau(n) = 6$. This implies the prime factorization of n must be $n = p^5$ or $n = p^2 q$ for distinct primes p, q. If $n^2 = p^{10} \leq 2019$ then $p = 2$. If $n = p^2 q$ then $p^4 \leq 2019$ so $p < 7$. If $p = 2$ then $q^2 < 127$ and hence, $q \leq 11$ yielding five solutions. If $p = 3$, then $q^2 < 25$ so two solutions. Finally, if $p = 5$ the there are two solutions as well. In total, we have found 10 solutions.

34. Let $f(x)$ be a rational coefficient polynomial with $(1+i)\frac{\sqrt{6}}{4} + (1-i)\frac{\sqrt{2}}{4}$ as a root. Find the minimum possible positive degree of $f(x)$.
The complex number given is the 24^{th} root of unity. The minimum possible rational coefficient polynomial is the cyclotomic polynomial, which has degree $\varphi(24) = 8$.

35. Find all real numbers $x \in \left[0, \frac{\pi}{2}\right]$, such that $(2 - \sin 2x)\sin(x + \frac{\pi}{4}) = 1$.
Let $a = \sin(x + \frac{\pi}{4})$. If $x \in [0, \frac{\pi}{2}]$, then $\frac{\sqrt{2}}{2} \leq a \leq 1$ and $2 - \sin 2x = 3 - 2a^2$. Therefore $2a^3 - 3a + 1 = 0$. One root is $a = 1$ or $x = \frac{\pi}{4}$. Other roots are $2a^2 + 2a - 1 = 0 \to a = \frac{-1 \pm \sqrt{3}}{2}$. But $\frac{\sqrt{3}-1}{2} < \frac{\sqrt{2}}{2}$.

36. An integer $n \geq 2$ is called friendly if there exists a family $A_1, A_2, \ldots, A_n$ of subsets of the set $\{1, 2, \ldots, n\}$ such that:
(1) $i \notin A_i$ for every $i = 1, 2, \ldots, n$;
(2) $i \in A_j$ if and only if $j \notin A_i$, for every distinct $i, j \in \{1, 2, \ldots, n\}$;
(3) $A_i \cap A_j$ is non-empty, for every $i, j \in \{1, 2, \ldots, n\}$.
Determine the smallest friendly number.
We claim the smallest friendly number is $n = 7$. For $n = 7$, let $A_i = \{i+1, i+2, i+4\} \pmod 7$; it satisfies (1), (2), and we note that $A_i \cap A_{i+3} = i+4, A_i \cap A_{i+5} = i+2, A_i \cap A_{i+6} = i+1$, satisfying (3). For $n < 7$, condition (2) implies that $\sum_{i=1}^n |A_i| = \binom{n}{2}$. Then $(\sum_{i=1}^n |A_i|)/n = \binom{n}{2}/n = \frac{n-1}{2} < 3$, so there exists some \$ A_i\$ which contains only two elements (say $A_i = \{j, k\}$). But then by condition (2), we have that k cannot belong to A_j and j belong to A_k simultaneously, so either $A_i \cup A_j$ or $A_i \cup A_j$ is the empty set, contradiction.

37. On the Cartesian plane the curve (C) has equation $x^2 = y^3$. A line d varies on the plane such that d always cut (C) at three distinct points with x-coordinates x_1, x_2, x_3. Determine the maximum possible value of $\sqrt[3]{\frac{x_1x_2}{x_3^2}} + \sqrt[3]{\frac{x_2x_3}{x_1^2}} + \sqrt[3]{\frac{x_3x_1}{x_2^2}}$.
Let d be the line $y = ax+b$. Then, x_1, x_2, x_3 are the roots of the following equation: $0 = (ax+b)^3 - x^2 = a^3x^3 + (3a^2b - 1)x^2 + 3ab^2x + b^3$. Since x_1, x_2, x_3 are distinct, we have $a \neq 0$. We can simplify the desired expression as $S = \sqrt[3]{\frac{x_1x_2}{x_3^2}} + \sqrt[3]{\frac{x_2x_3}{x_1^2}} + \sqrt[3]{\frac{x_3x_1}{x_2^2}} = \sqrt[3]{x_1x_2x_3}\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right)$. By Vieta, $S = \frac{x_2x_3 + x_1x_3 + x_1x_2}{\sqrt[3]{(x_1x_2x_3)^2}} = \frac{3b^2/a^2}{\sqrt[3]{\left(-\frac{b^3}{a^3}\right)^2}} = 3$. Therefore, the maximum possible value of our expression is3.

38. Let $f(x,y) = \frac{x+y}{(x^2+1)(y^2+1)}$. Find the maximum value of $f(x,y)$ for all $x, y \in \mathbb{R}$.
Let $x = \tan\theta, y = \tan\phi$ and then $f = \frac{\tan\theta + \tan\phi}{\sec^2\theta\sec^2\phi} = \cos\theta\cos\phi(\sin\theta\cos\phi + \sin\phi\cos\theta) = \cos\theta\cos\phi\sin(\theta + \phi)$ By AM-GM Inequality, we have $f \leq \left(\frac{\cos\theta + \cos\phi + \sin(\theta+\phi)}{3}\right)^3$. Let $P = \cos\theta + \cos\phi + \sin(\theta + \phi)$, and from product to sum, we have $P = 2\cos(\frac{\theta+\phi}{2})\left[\cos(\frac{\theta-\phi}{2}) + \sin(\frac{\theta+\phi}{2})\right] \leq 2\cos(\frac{\theta+\phi}{2})\left[1 + \sin(\frac{\theta+\phi}{2})\right]$. Letting $z = \sin(\frac{\theta+\phi}{2})$, we get the cleaner expression $P^2 \leq 4(1-z^2)(1+z)^2$. Again by AM-GM,

$$P^2 \leq 4 \cdot 27(1-z)\left(\frac{1+z}{3}\right)^3 \leq \frac{4 \times 27}{4^4}\left[1 - z + 3 \cdot \frac{1+z}{3}\right]^4 = \frac{27}{4}$$

or $P \leq \frac{3\sqrt{3}}{2}$. Note equality occurs when $\cos\theta = \cos\phi = \sin(\theta + \phi)$ and $\sin(\frac{\theta+\phi}{2}) = \frac{1}{2}$. That is, when $\theta = \phi = \frac{\pi}{6}$ which yields $f_{\max} = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$ attained when $x = y = \frac{1}{\sqrt{3}}$.

39. For any $x_i \geq 0$, $i = 1, 2, \ldots, 2018$, if let $x_{2019} = x_1$, find the minimum value of

$$\sum_{k=1}^{2018} \sqrt{\frac{1}{(x_k+1)^2} + \frac{x_{k+1}^2}{(x_{k+1}+1)^2}}$$

Let $x_k = \tan^2 \theta_k$ for $k = 1, 2, \ldots, 2018$, where $\theta_k \in \left[0, \frac{\pi}{2}\right)$, and take $\theta_{n+1} = \theta_1$, then

$$\sqrt{\frac{1}{(x_k+1)^2} + \frac{x_{k+1}^2}{(x_{k+1}+1)^2}} = \sqrt{\cos^4 \theta_k + \sin^4 \theta_{k+1}} \geq \sqrt{\frac{1}{2}\left(\cos^2 \theta_k + \sin^2 \theta_{k+1}\right)^2} = \frac{\cos^2 \theta_k + \sin^2 \theta_{k+1}}{\sqrt{2}}$$

Thus,

$$\sum_{k=1}^{2018} \sqrt{\frac{1}{(x_k+1)^2} + \frac{x_{k+1}^2}{(x_{k+1}+1)^2}} \geq \sum_{k=1}^{2018} \frac{\cos^2 \theta_k + \sin^2 \theta_{k+1}}{\sqrt{2}} = \frac{2018}{\sqrt{2}} = 1009\sqrt{2}$$

40. Find the maximum number of edges of a 4 dimensional cube that are cut by a hyperplane.
Without loss of generality, assume that the cube is an unit cube with vertices (a, b, c, d) where $a, b, c, d \in \{0, 1\}$. Consider the hyperplanes

$$H_k : x_1 + x_2 + \cdots + x_4 = k + \frac{1}{2}, 0 \leq k \leq 3$$

Then H_k crosses every edge joining one of the $\binom{4}{k}$ points with k nonzero coordinates to its $4 - k$ neighbors with $k+1$ nonzero coordinates and only these edges. Thus, such a plane crosses $(n-k)\binom{n}{k}$. This is maximized when $k = 2$, which implies that the answer is 12.