Convolution Method for the Complete Response of a Series Ł-Ɍ Network Connected to an Excitation Source of Sinusoidal Potential

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Abstract- The analysis of electric networks containing energy storage elements like a capacitor or an inductor or both a capacitor and an inductor is an essential course for most of the branches of the engineering. The response of such networks is generally obtained by adopting the classical method or by algebraic and analytic methods. This paper presents a convolution method for obtaining the complete response of a series electric network of two passive elements namely an inductor of inductance Ł and a resistor of resistance Ɍ (i.e. a series Ł-Ɍ network), connected to an excitation source of sinusoidal potential. The response obtained by solving the governing differential equation will provide an expression for the electric current which flows in the series Ł-Ɍ network connected to an excitation source of sinusoidal potential. This paper presents a new approach to demonstrate the use of the convolution in obtaining the complete response of a series ŁɌ network through the application of the convolution method. The response obtained by solving the governing differential equation by the application of the convolution method will provide an expression for the electric current. In this paper, the response of a series Ł-Ɍ network is provided as a demonstration of the application of the convolution method.

Index Terms- Convolution; Current; Series Electric Network; Response.

I. INTRODUCTION

Active electric elements are defined as those which have the ability to deliver average electric power greater than zero to the external electric devices in an infinite time interval whereas, Passive electric elements are defined as those which do not have the ability to do so. The electric circuit of a series ŁɌ network consists of two passive electric elements namely an inductor Ł and a resistor Ɍ, connected in series to an active electric element namely an excitation source of sinusoidal potential. It is used as a tuning circuit, which is an example of band pass filtering, or resonant circuit in the radio and television sets to tune or resonate a particular frequency band from the wide range of radio frequency components, or in the chokes of luminescent tubes [1-4].

II. LAPLACE TRANSFORMATION

The Laplace transformation of a function g(y), where y ≥ 0, is denoted by $G(q)$ or L {g(y)} and is defined as L{g(y)} = $G(q) = \int_0^{\infty} e^{-qy} g(y)dy$, provided that the integral exists, where q is the parameter which may be a real or complex number and L is the Laplace transform operator. The Laplace Transformation of some elementary functions are written as [5-8]

- $L\{1\} = \frac{1}{q}$, $q > 0$
- $L\{y^n\} = \frac{n!}{q^{n+1}}$,
  $where\ n = 0,1,2,3 \ldots\ldots\ldots$
- $L\{e^{cy}\} = \frac{1}{q-c}$, $q > c$
- $L\{sincy\} = \frac{c}{q^2+c^2}$, $q > 0$
- $L\{sinhcy\} = \frac{c}{q^2-c^2}$, $q > |c|$
- $L\{coscy\} = \frac{q}{q^2+c^2}$, $q > 0$
- $L\{coshcy\} = \frac{q}{q^2-c^2}$, $q > |c|$

A. Laplace Transformation of Derivative of a function

If the function g(y), where y ≥ 0, is having an exponential order, that is if g(y) is a continuous function and is a piecewise continuous function on any interval, then the Laplace transform of derivative of g(y) i.e. $L\{g'(y)\}$ is given by [5-8]

$$L\{g'(y)\} = \int_0^{\infty} e^{-qy}\, g'(y)dy$$

Integrating by parts, we get

$L\{g'(y)\}$= $[0 - g(0)] - \int_0^{\infty} -qe^{-qy}\, g(y)dy$,

Or $L\{g'(y)\} = -g(0) + q\int_0^{\infty} e^{-qy}\, g(y)dy$

Or $L\{g'(y)\} = qL\{g(y)\} - g(0)$

Or $L\{g'(y)\} = qG(q) - g(0)$

Now, since $L\{g'(y)\} = qL\{g(y)\} - g(0)$,

Therefore, $L\{g''(y)\} = qL\{g'(y)\} - g'(0)$

Or $L\{g''(y)\} = q\{qL\{g(y)\} - g(0)\} - g'(0)$

Or $L\{g''(y)\} = q^2L\{g(y)\} - qg(0) - g'(0)$

Or $L\{g''(y)\} = q^2G(q) - qg(0) - g'(0)$, and so on.

B. Inverse Laplace Transformation

The inverse Laplace transform of the function $G(q)$ is denoted by $L^{-1}[G(q)]$ or g(y). If we write L [g(y)] = $G(q)$, then $L^{-1}[G(q)]$ = g(y), where L^{-1} is called the inverse Laplace transform operator. The Inverse Laplace Transformations of some functions are written as [5-8]

- $L^{-1}\{\frac{1}{q}\} = 1$
- $L^{-1}\{\frac{1}{(q-c)}\} = e^{cy}$
- $L^{-1}\{\frac{1}{q^2+c^2}\}= \frac{1}{c}\sin cy$
- $L^{-1}\{\frac{q}{q^2+c^2}\} = \cos cy$
- $L^{-1}\{\frac{q}{q^2-c^2}\} = \cos hcy$
- $L^{-1}\{\frac{1}{q^2-c^2}\} = \frac{1}{c}\sin hcy$
- $L^{-1}\{\frac{1}{q^n}\} = \frac{y^{n-1}}{(n-1)!}$, n > 0.

I. CONVOLUTION AND CONVOLUTION THEOREM

The convolution of two functions $\varphi(\gamma)$ and $\emptyset(\gamma)$ which are defined and piecewise continuous in $[0,\infty)$, is denoted by $(\varphi * \emptyset)(\gamma)$ and is defined as [5-8]

$(\varphi * \emptyset)(\gamma) = \int_0^{\gamma} \varphi(r)\, \emptyset(\gamma - r)\, dr$, where $\gamma \geq 0$.

If these functions $\varphi(\gamma)$ and $\emptyset(\gamma)$ are of exponential order, then the Laplace transform of $[(\varphi * \emptyset)(\gamma)]$ is given by

$Ł[(\varphi * \emptyset)(\gamma)] = Ł[\varphi(\gamma)]\, Ł[\emptyset(\gamma)] = \bar{\varphi}(q)\, \bar{\emptyset}(q)$,

where $\bar{\varphi}(q)$ and $\bar{\emptyset}(q)$ are Laplace transforms of $\varphi(\gamma)$ and $\emptyset(\gamma)$ and Ł is Laplace transform operator.

A. Proof of convolution theorem

We can write

$Ł[\varphi(\gamma)]\, Ł[\emptyset(\gamma)] = \bar{\varphi}(q)\, \bar{\emptyset}(q)$

$$= \int_0^{\infty} e^{-qr}\, \varphi(r) dr \int_0^{\infty} e^{-q\delta}\, \emptyset(\delta) d\delta$$

$$= \int_0^{\infty}\int_0^{\infty} e^{-q(r+\delta)} \varphi(r)\emptyset(\delta) dr\, d\delta$$

$$= \int_0^{\infty} \varphi(r) dr \int_0^{\infty} e^{-q(r+\delta)} \emptyset(\delta) d\delta$$

Let us put $r + \delta = \gamma$, where r is fixed, then $\delta = \gamma - r$ and the value of γ vary from r to ∞.

Hence we write

$Ł[\varphi(\gamma)]\, Ł[\emptyset(\gamma)] = \bar{\varphi}(q)\, \bar{\emptyset}(q) = \int_0^{\infty} \varphi(r) dr \int_r^{\infty} e^{-q\gamma} \emptyset(\gamma - r) d\gamma$.

On changing the order of integration, we can write the order of integration as $0 \leq \gamma \leq \infty$ and $0 \leq r \leq \gamma$. Therefore,

$Ł[\varphi(\gamma)]\, Ł[\emptyset(\gamma)] = \bar{\varphi}(q)\, \bar{\emptyset}(q) = \int_0^{\infty} e^{-q\gamma} dr \int_0^{\gamma} \varphi(r)\emptyset(\gamma - r) dr$

$= \int_0^{\infty} e^{-q\gamma} [\int_0^{\gamma} \varphi(r)\emptyset(\gamma - r) dr]\, d\gamma$

$= \int_0^{\infty} e^{-q\gamma} [(\varphi * \emptyset)(y)]\, d\gamma$

$= Ł[(\varphi * \emptyset)(y)]$

Hence we can write

$\bar{\varphi}(q)\, \bar{\emptyset}(q) = Ł[(\varphi * \emptyset)(\gamma)]$

Applying inverse Laplace Transform, we can write

$Ł^{-1}[\bar{\varphi}(q)\, \bar{\emptyset}(q)] = [(\varphi * \emptyset)(\gamma)]$

II. FORMULATION

A. Governing differential equation

We will take a series Ł-R network to which a sinusoidal excitation voltage source of potential V = $V_0 \sin \omega t$ is applied through a key K as shown in figure 1.

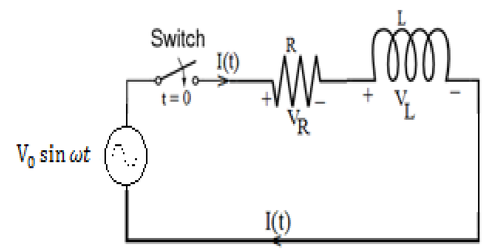

Figure 1: Series LR network with sinusoidal voltage source.

As the switch is closed at t = 0, the potential drops across the network elements are given by [1-2]

$V_R(t) = I(t)R,\ V_L(t) = ŁĐ_t[I(t)]$.

Therefore, the application of Kirchhoff's loop law to the loop shown in figure 2 provides

$V_R(t) + V_Ł(t) = V$

Or

$R\, I(t) + ŁĐ_t[I(t)] = V_0 \sin \omega t$….. (1) $Đ_t \equiv \frac{d}{dt}$.

Differentiate equation (1), we get a linear homogeneous differential equation of order 2 as given below:

$RĐ_t[I(t)] + ŁĐ_t^2[I(t)] = V_0\, \omega \cos \omega t$

Or

$ŁĐ_t^2[I(t)] + RĐ_t[I(t)] = V_0\, \omega \cos \omega t$

Or

$Đ_t^2[I(t)] + \frac{R}{Ł} Đ_t[I(t)] = \frac{V_0 \omega}{Ł} \cos \omega t$ …. (2)

B. Solution of governing differential equation

To solve equation (2), we first write the relevant boundary conditions [1-4] as follows:

- Since the current through the inductor and the electric potential across the capacitor cannot be changed instantaneously, therefore, as the switch is closed at the instant t = 0, then I (0) = 0.
- Since at the instant t = 0, I (0) = 0, therefore, equation (1) provides $ŁĐ_t[I(0)] = 0$ or $Đ_t[I(0)] = 0$.

The Laplace transform of equation (2) provides

$q^2\bar{I}(\mathrm{q}) - qI(0)$ -$D_t[I(0)] + \frac{\mathrm{R}}{\text{Ł}}\{q\,\bar{I}(\mathrm{q}) - I(0)\}$
$= \frac{\mathrm{V_0}\omega}{\text{Ł}}\cos\omega t$....... (3)

Applying boundary conditions: $I(0) = 0$ and $\text{Đ}_t[\mathrm{I}(0)] = 0$, equation (3) becomes,

$$q^2\bar{I}(\mathrm{q}) + \frac{\mathrm{R}}{\text{Ł}}q\,\bar{I}(\mathrm{q}) = \frac{\mathrm{V_0}\omega}{\text{Ł}}\frac{q}{q^2+\omega^2}$$

Or

$$\bar{I}(\mathrm{q})\,[q^2 + \frac{\mathrm{R}}{\text{Ł}}q\,] = \frac{\mathrm{V_0}\omega}{\text{Ł}}\frac{q}{q^2+\omega^2}$$

Or

$$\bar{I}(\mathrm{q}) = \frac{\mathrm{V_0}}{\text{Ł}}\frac{\omega}{q^2+\omega^2}\left[\frac{1}{q+\frac{\mathrm{R}}{\text{Ł}}}\right] \ldots\ldots\ldots\ldots (4)$$

Let F (q) = $\frac{\omega}{q^2+\omega^2}$ and G (q) = $\frac{1}{q+\frac{\mathrm{R}}{\text{Ł}}}$,

Then the inverse Laplace transforms of these functions are given by f (t) = $\sin\omega t$ and g (t) = $e^{-\frac{\mathrm{R}}{\text{Ł}}t}$.

Equation (4) can be rewritten as

$\bar{I}(\mathrm{q}) = \frac{\mathrm{V_0}}{\text{Ł}}$ [F (q) x G (q)] ………… (5)

Taking inverse Laplace transform of equation (5), we can write

$I(t) = \frac{\mathrm{V_0}}{\text{Ł}}L^{-1}$[F (q) x G (q)] …….. (6)

Now applying convolution theorem, we can write

L^{-1}[F (q) x G (q)] = (f * g) (t) = $\int_0^t f(\tau)g(t-\tau)\,d\tau$

Or L^{-1}[F (q) x G (q)] = $\int_0^t \sin\omega\tau\;e^{-(\frac{\mathrm{R}}{\text{Ł}})(t-\tau)}\,d\tau$

Or L^{-1}[F (q) x G (q)] $= e^{-(\frac{\mathrm{R}}{\text{Ł}})t}\int_0^t e^{\frac{\mathrm{R}}{\text{Ł}}\tau}\sin\omega\tau\,d\tau$ …….. (7)

Using identity:

$\int e^{bx}\sin ax\;dx = e^{bx}\left[\frac{b\sin ax - a\cos qx}{a^2+b^2}\right]$, we can write equation (7) as

L^{-1}[F (q) x G (q)] =

$$e^{-(\frac{\mathrm{R}}{\text{Ł}})t}\left\{e^{\frac{\mathrm{R}}{\text{Ł}}\tau}\left[\frac{\frac{\mathrm{R}}{\text{Ł}}\sin\omega\tau-\omega\cos\omega\tau}{(\omega)^2+(\frac{\mathrm{R}}{\text{Ł}})^2}\right]\right\}_0^t$$

Or

L^{-1}[F (q) x G (q)] =

$$e^{-(\frac{\mathrm{R}}{\text{Ł}})t}\left\{e^{\frac{\mathrm{R}}{\text{Ł}}t}\left[\frac{\frac{\mathrm{R}}{\text{Ł}}\sin\omega t-\omega\cos\omega t}{(\omega)^2+(\frac{\mathrm{R}}{\text{Ł}})^2}\right]-\left[\frac{-\omega}{(\omega)^2+(\frac{\mathrm{R}}{\text{Ł}})^2}\right]\right\}$$

Or

L^{-1}[F (q) x G (q)] =

$$\left\{\left[\frac{\frac{\mathrm{R}}{\text{Ł}}\sin\omega t-\omega\cos\omega t}{(\omega)^2+(\frac{\mathrm{R}}{\text{Ł}})^2}\right]+\left[\frac{\omega}{(\omega)^2+(\frac{\mathrm{R}}{\text{Ł}})^2}e^{-(\frac{\mathrm{R}}{\text{Ł}})t}\right]\right\}$$

Or

L^{-1}[F (q) x G (q)] =

$$\left\{\left[\frac{\text{Ł}}{(\omega\text{Ł})^2+\mathrm{R}^2}\mathrm{R}\sin\omega t-\omega\text{Ł}\cos\omega t\right]+\left[\frac{\omega\text{Ł}^2}{(\omega\text{Ł})^2+\mathrm{R}^2}e^{-(\frac{\mathrm{R}}{\text{Ł}})t}\right]\right\}\ldots\ldots (8)$$

Using equation (8) in equation (6), we get

$I(t) =$

$$\frac{\mathrm{V_0}}{\text{Ł}}\left\{\left[\frac{\text{Ł}}{(\omega\text{Ł})^2+\mathrm{R}^2}(\mathrm{R}\sin\omega t-\omega\text{Ł}\cos\omega t)\right]+\left[\frac{\omega\text{Ł}^2}{(\omega\text{Ł})^2+\mathrm{R}^2}e^{-(\frac{\mathrm{R}}{\text{Ł}})t}\right]\right\}$$

Or

$I(t) =$

$$\left\{\left[\frac{\mathrm{V_0}}{(\omega\text{Ł})^2+\mathrm{R}^2}(\mathrm{R}\sin\omega t-\omega\text{Ł}\cos\omega t)\right]+\left[\frac{\mathrm{V_0}\omega\text{Ł}}{(\omega\text{Ł})^2+\mathrm{R}^2}e^{-(\frac{\mathrm{R}}{\text{Ł}})t}\right]\right\}$$

Or

$I(t) =$

$$\frac{\mathrm{V_0}}{\sqrt{(\omega\text{Ł})^2+\mathrm{R}^2}}\left\{\left[\left(\frac{\mathrm{R}}{\sqrt{(\omega\text{Ł})^2+\mathrm{R}^2}}\sin\omega t-\frac{\omega\text{Ł}}{\sqrt{(\omega\text{Ł})^2+\mathrm{R}^2}}\cos\omega t\right)\right]+\left[\frac{\omega\text{Ł}}{\sqrt{(\omega\text{Ł})^2+\mathrm{R}^2}}e^{-(\frac{\mathrm{R}}{\text{Ł}})t}\right]\right\}\ldots\ldots (9)$$

Let us put $\frac{\omega\text{Ł}}{\sqrt{(\omega\text{Ł})^2+\mathrm{R}^2}} = \sin\emptyset$ and $\frac{\mathrm{R}}{\sqrt{(\omega\text{Ł})^2+\mathrm{R}^2}} = \cos\emptyset$ such that $\tan\emptyset = \frac{\omega\text{Ł}}{\mathrm{R}}$, then we can rewrite equation (9) as

$$I(t) = \frac{\mathrm{V_0}}{\sqrt{(\omega\text{Ł})^2+\mathrm{R}^2}}\left\{[(\cos\emptyset\sin\omega t-\sin\emptyset\cos\omega t)]+\left[\sin\emptyset\,e^{-(\frac{\mathrm{R}}{\text{Ł}})t}\right]\right\}$$

Or

$$I(t) = \frac{\mathrm{V_0}}{\sqrt{(\omega\text{Ł})^2+\mathrm{R}^2}}\left\{\sin(\omega t-\emptyset)+\sin\emptyset\,e^{-(\frac{\mathrm{R}}{\text{Ł}})t}\right\}\ldots\ldots (10)$$

This equation (10) provides an expression for the complete response (electric current flowing through a series $\text{Ł}-\mathrm{R}$ network) a series $\text{Ł}-\mathrm{R}$ network connected to an excitation source of sinusoidal potential V= $\mathrm{V_0}\sin\omega t$.

V. CONCLUSION

In this paper, an attempt is made to exemplify the convolution approach for determining the complete response (electric current) of a series Ł-R network connected to an excitation source of sinusoidal potential through the application of the convolution method. This approach brings up the convolution approach as a powerful technique for determining the response of electronic circuits by solving their governing differential equations via the convolution method.

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