International Journal of Modern Science and Technology *Vol. 2, No. 10, 2017. Page 341-344.*
http://www.ijmst.co/ ***ISSN: 2456-0235***.

Research Article

Numerical Range of Maximal Jordan Elementary Operator

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Abstract

Studies on numerical ranges such as essential numerical ranges, spatial numerical ranges, algebraic numerical ranges and maximal numerical ranges for elementary operators have been considered by different scholars. In this paper we focus on maximal numerical range for a Jordan elementary operator. The results in this paper show that if H is an infinite dimensional complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H, then the maximal numerical range $M(U)$ of a Jordan elementary operator is nonempty, closed and convex. Furthermore we show that the maximal numerical radius is equivalent to the norm of Jordan elementary operator but are not necessarily equal.

Keywords: Numerical range; Maximal numerical range; Numerical radius; Norm and Jordan elementary operator.

Introduction

Studies on properties of different numerical ranges such as essential, spatial, algebraic and maximal numerical ranges have been considered on general elementary operators [1-4]. Such properties as linearity, closure, compactness, non-emptiness, convexity have been generally considered on elementary operators [5]. In this paper however, we consider maximal numerical range of a Jordan elementary operator which is an example of elementary operator and indeed we show that it is non-empty, closed and convex and show that the maximal numerical radius is equivalent to the norm of Jordan elementary operator but are not necessarily equal [6]. In section 1, we give a brief introduction, section 2 we give methodology and basic concepts, section 3 has got the results and discussions while section 4 gives the conclusion.

Research methodology

In writing this paper, basic knowledge of functional analysis and operator theory are essential. The methodology involved the use of known inequalities such Cauchy-Schwatz [7] and Minkowski's inequality [8]. We also involved the use of technical approaches such as tensor product [9] and direct sum decomposition [10]. Lastly we involved the use of known definitions and fundamental results as stated below.

Definition 2.1 [11, Section 2]. Let H be an infinite dimensional complex Hilbert space and $B(H)$ the C*-algebra of all bounded linear operators on H. The Jordan elementary operator (implemented by P,Q) is defined by $U_{P,Q}(T) = PTQ + QTP$; $\forall T \in B(H)$ $\forall T \in B(H)$ and P_i,Q_i fixed in $B(H)$.

Definition 2.2 [12, Definition 2.1] For an operator $U \in [B(H)]$, we define the numerical range (also known as the field of values) of U by $W(U) = \{\langle UT_n, T_n\rangle : T_n \in H, \|T_n\| = 1\}$

Definition 2.3 13, Definition 1.1.13**]** The maximal numerical range $M(\mathrm{U})$ is defined by $M(\mathrm{U}) = \{\lambda \in \mathrm{C} : \exists\{T_{n\geq 1}\} \in B(H) \text{ s.t } \|T_n\| = 1, \lambda = \lim_{n\to\infty}\langle UT_n, T_n\rangle, \lim_{n\to\infty}\|UT_n\| = \|U\|\}$.

Definition 2.4 [14, Definition 3.2.34**]** For a p-tuple, $U=(U_1,\ldots,U_p)$ of operators on a Hilbert space H, we define the numerical radius by $w(U)=\{(\langle U_1x,x\rangle,\ldots,\langle U_px,x\rangle): x\in H, \|x\|=1\}$.

Definition 2.5 [15, Definition 2.1]. A norm on a vector space V is a nonnegative real valued function
$\|.\| : V \to R$ satisfying the following properties:
i). $\|v\| \geq 0$ $\forall v \in V$
ii). $\|v\| = 0$ iff $v = 0$

Received: 20.09.2017; Received after Revision: 01.10.2017; Accepted: 01.10.2017; Published: 04.10.2017

iii). $\| \lambda v\| = | \lambda | \, \|v\| \;\; \forall \lambda \in \mathbf{C}$ and $v \in V$
iv). $\|v+w\| \leq \|v\| + \|w\|$

Results and discussion

In this section we prove that maximal numerical range of Jordan elementary operator is non-empty, closed and convex. We also show that the maximal numerical radius is equivalent to the norm of Jordan elementary operator but are not necessarily equal.

Remark 4.10. From definition 2.3 above, we see that $M(U)$ is contained in the closed disk of radius $\|U\|$ centred at the origin, $M(\lambda I_H) = \{\lambda\}$ and that $M(\lambda U) = \lambda M(U)$ $\forall \lambda \in C$ and $U \in B(H)$ and moreover $M(U) \subseteq \overline{M(U)}$[16].

Proposition 4.11. Let H be a complex Hilbert space and $U : B(H) \rightarrow B(H)$ and let $U \in [B(H)]$, where $[B(H)]$ is the space of all bounded linear operators, then the following conditions hold:
(i). $M(U)$ is nonempty
(ii). $M(U)$ is closed

Proof. (i). By the definition of operator norm, there exists a sequence of contractive vector spaces $\{T_n\}_{n\geq 1}$ such that$\lim_{n\rightarrow\infty}\|UT_n\| = \|U\|$ and since $\{\langle UT_n, \underline{T}_n\rangle\}_{n\geq 1}$ is a bounded sequence [17], then $\lim_{n\rightarrow\infty}\langle UT_n, T_n\rangle \rightarrow \lambda$But $\lambda \in M(U)$ which implies that $M(U) \neq \emptyset$
(ii). Let $\{\lambda_n\}_{n\geq 1}$ be a sequence in $M(U)$ that converges to some $\lambda \in C$. By definition of maximal numerical range, for each $n \in N$, there exists a contractive vector space T_n such that $\|U\| \leq \|UT_n\| + \frac{1}{n}$ and $|\lambda_n - \langle UT_n, T_n\rangle| < \frac{1}{n}$. Thus $\lim_{n\rightarrow\infty} \|UT_n\| = \|U\|$ and $\lim_{n\rightarrow\infty}\langle UT_n, T_n\rangle = \lambda$ hence $\lambda \in M(U)$. Thus $M(U)$ is closed.

Lemma 4.12. Let $U \in [B(H)]$ and let $\lambda \in C$. Then $M(U) = \{\lambda\}$ if and only if $U = \lambda I_H$

Proof. By Remark 4.5, we have that $M(\lambda I_H) = \{\lambda\}$ for any $\lambda \in C$. Suppose $M(U) = \{\lambda\}$ for some $U \in B(H)$. Then for all $T \in H$ with $\|T_n\| = \|T\| = 1$ $\langle(\lambda I_H - U)T_n, T_n\rangle = \lambda - \langle UT_n, T_n\rangle = \lambda - \lambda = 0$. Hence $\langle(\lambda I_H - U)T_n, T_n\rangle = 0$ for $T \in H$ and Thus $\lambda I_H - U = 0$.

Theorem 4.13. Let $U \in [B(H)]$. Then $\lambda \in M(U)$ if and only if there exists an orthonormal sequence $(T_n)_{n\geq 1}$ such that $\lim_{n\rightarrow\infty}\langle UT_n, T_n\rangle = \lambda$

Proof. Let there exists an orthonormal sequence $(T_n)_{n\geq 1}$ such that $\lim_{n\rightarrow\infty}\langle UT_n, T_n\rangle = \lambda$. Then, for every compact operator $V \in B(H)$, $\lambda = \lim_{n\rightarrow\infty}\langle UT_n, T_n\rangle = \lim_{n\rightarrow\infty}\langle(U + V)T_n, T_n\rangle \in \overline{M(U + V)}$ and since V was picked arbitrary, it implies that, $\lambda \in M(U)$. Now let $\lambda \in M(U)$. Therefore, $\lambda \in \overline{M(U)}$so there exists a contractive vector space T_1 such that
$|\langle UT_1, T_1\rangle - \lambda| \leq \frac{1}{2}$
Let $L := \text{span}\{T_1\}$, let D_1 be the orthogonal projection onto L, let $\mu \in M((I_H - D_1)U \,|L_1^{\perp})$, and let $F_1 := \mu_1 D_1 - D_1UD_1 - D_1U(I_H - D_1) - (I_H - D_1)UD_1$. This shows that F_1 is a finite rank operator on H and thus is compact. Therefore, $\lambda \in \overline{M(U + F1)} = \overline{M(\mu 1D1 + (IH - D1)U(IH - D1))}$
But
$M(\mu_1D_1 + (I_H - D_1)U(I_H - D_1)) = \{\langle(\mu_1D_1 + (I_H - D_1)U(I_H - D_1))\eta, \eta\rangle | \eta \in H, \|\eta\| = 1\}$
$= \{\mu_1\|\eta_1\|^2 + \langle(I_H - D_1)U(I_H - D_1)\eta_2, \eta_2\rangle | \eta_1 \in L_1, \eta_2 \in L_1^{\perp}, \|\eta_1\|^2 + \|\eta_2\|^2 = 1\}$
$= \{\mu_1\|\eta_1\|^2 + \|\eta_2\|^2\langle(I_H - D_1)U(I_H - D_1)\frac{1}{\|\eta 2\|}\eta_2, \frac{1}{\|\eta 2\|}\eta_2\rangle | \eta_1 \in L_1, \eta_2 \in L_1^{\perp}, \|\eta_1\|^2 + \|\eta_2\|^2 = 1\}$ and since $\mu_1 \in M((I_H - D_1)U \,|_{L1\perp})$ and that $M((I_H - D_1)U|_{L1\perp})$ is convex, we obtain that $M(\mu_1D_1 + (I_H - D_1)U(I_H - D_1)) = M((I_H - D_1)U|_{L1\perp})$. Hence $\lambda \in M((I_H - D_1)U|_{L1\perp})$ so there exists a contractive vector space $T_2 \in L_1^{\perp}$ (i.e T_2 is orthogonal to T_1) such that$|\langle UT_2, T_2\rangle - \lambda| \leq \frac{1}{2^2}$
Now suppose $T_1, \ldots, T_n$ are orthonormal contractive vector spaces such that$|\langle UT_n, T_n\rangle - \lambda| \leq \frac{1}{2^n}$ we can repeat the above procedure with $L_n := span\{T_1, \ldots, T_n\}$,$D_n$ the orthogonal projection onto L_n, $\mu_n \in M((I_H - D_1)U|_{L1\perp})$, and $F_n := \mu_nD_n - D_nUD_n - D_nU(I_H - D_n) - (I_H - D_n)UD_n$ to obtain the vector space T_{n+1} orthogonal to each T_k for $1 \leq k \leq n$ such that $|\langle UT_{n+1}, T_{n+1}\rangle - \lambda| \leq \frac{1}{2^{n+1}}$. Hence by recursion, there exists an orthonormal sequence $(T_n)_{n\geq 1}$ in H such that $\lim_{n\rightarrow\infty}\langle UT_n, T_n\rangle = \lambda$ as required.

Theorem 4.14. The maximal numerical range $M(U)$ of $U \in [B(H)]$ is convex

Proof. Suppose $\lambda_1, \lambda_2 \in M(U)$ are given, then we need to show that $(1 - b)\lambda_1 + b\lambda_2 \in M(U)$ whenever $b \in [0, 1]$. Now if $C = \alpha I + \beta U$, where $\alpha, \beta \in C$ are such that $0 = \alpha + \beta\lambda_1$ and $1 = \alpha + \beta\lambda_2$, it suffices to prove that $b \in M(C)$ for all $b \in [0, 1]$. We fix unit vectors $x, y \in H$ such that $0 = \langle Cx_n, x_n\rangle$, $1 = \langle Cy_n, y_n\rangle$ and define $g : R \rightarrow C$ by $g(b) = \langle Cx_n, y_n\rangle\, exp(-ib) + \langle Cy_n, x_n\rangle\, exp(ib)$ $b \in R$. Since $\cos \pi = -1$, it implies that $g(b + \pi) = -g(b)$ for every $b \in R$. Moreover, there exists $b_0 \in [0, \pi]$ such that $Img(b_0) = 0$. Since $Img(0) = -Img(\pi)$ and g is a continuous function, there exists $b_0 \in [0, \pi]$ such that $Img(b_0) = 0$. Now fix the vectors x, $\hat{y} = exp(ib_0)y$ are linearly independent. Otherwise, $x = \hat{y}\alpha$ for some $\alpha \in C$, $|\alpha| = 1$ and $0 = \langle Cx_n, x_n\rangle = |\alpha|2\langle C\, \hat{y}_n, \hat{y}_n\rangle = \langle Cy_n, y_n\rangle = 1$. To complete the proof, define continuous functions z and f by $z(c) = \frac{(1-c)x + c\hat{y}}{\|(1-c)x + c\hat{y}\|}$, $c \in [0, 1]$ and $f(s) = \langle Cz(c), z(c)\rangle$, $c \in [0, 1]$. Which implies that f is a real-valued function with $f(0) = 0$ and $f(1) = 1$. Thus $b \in [0, 1] \subset f[0, 1] \subset [0, 1] \subset M(C)$, implying convexity.

Theorem 4.15. Let $U : B(H) \to B(H)$ *and* $T \subseteq U$ such that $T \in B(H)$ then maximal numerical radius, $m(T) \equiv \|T\|$.

Proof. Considering remark 4.10 that the maximal numerical range of Jordan elementary operator is contained in the closed disk of radius $\|U\|$ centred at the origin, and by Proposition 4.11, we establish that the maximal numerical radius of Jordan elementary operator is equivalent to its norm but are not necessarily equal as the following example illustrates.

Example 4.16. Consider $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in V_2(C)$ Then $M(T)$ is a closed disk of radius $\frac{1}{2}$ centred at the origin. To see this, we need to show that $\xi \in C_2$ is a unit vector if and only if we can write $\xi = (\cos(\theta)e^{i\theta 1}, \sin(\theta)e^{i\theta 2})$ for some $\theta_j \in [0, 2\pi)$ and $\theta \in [0, \frac{\pi}{2}]$. However, $\langle T(\cos(\theta)e^{i\theta 1}, \sin(\theta)e^{i\theta 2}), (\cos(\theta)e^{i\theta 1}, \sin(\theta)e^{i\theta 2})\rangle = \cos(\theta)\sin(\theta)e^{i(\theta 2-\theta 1)}$. By ranging all possible $\theta j \in [0, 2\pi)$ and $\theta \in [0, \frac{\pi}{2}]$ and using the fact that the range of $\cos(\theta)\sin(\theta) = \frac{1}{2}\sin 2(\theta)$ over $\theta \in [0, \frac{\pi}{2}]$ is $[0, \frac{1}{2}]$. We see that $M(T)$ is precisely the closed disk of radius $\frac{1}{2}$ centred at the origin. This shows that $m(T) = \frac{1}{2} \neq 1 = \|T\|$, thus demonstrating that maximal numerical radius and Jordan elementary operator norm are not equal.

Conclusions

Studies on properties of elementary operators have been of great concern to many mathematicians. Properties such as Numerical ranges, Spectrum, Compactness and Positivity have been studied with excellent results obtained. The norm property has remained an interesting area. Although the upper estimates of these norms are trivially obtainable in terms of their coefficients, estimating them from below has proved to be a challenge with different results being obtained. Entanglement is a basic physical resource to realize various quantum information and quantum communication tasks such as quantum cryptography, teleportation, dense coding and key distribution. In this paper, we have successfully proved that the maximal numerical range of a Jordan elementary operator is closed non-empty and convex which is in total agreement with the previous results worked on the elementary operators in general. We have further demonstrated that the maximal numerical radius and Jordan elementary operator norm are not equal. Moreover, it is shown that zero is in the numerical range.

Conflicts of Interest

Authors declare no conflict of interest.

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