Math 6345 – A. ODEs

Ruling out Closed Orbits – Nonlinear Dynamics & Chaos – Strogatz

Gradient systems

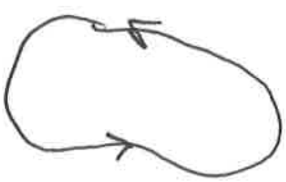
Suppose that we can write our system as

$$\dot{\vec{x}} = -\nabla V$$

for some $V(x,y) \in C^1(\mathbb{R}^2)$. Then closed orbits are impossible. (Note: we need V single valued)

Proof: By contradiction.

Suppose there is a closed orbit. Then after 1 circuit $\Delta V = 0$



However $\Delta V = \int_0^T dV = \int_0^T \frac{dV}{dt} \cdot dt$

$$= \int_0^T (V_x \dot{x} + V_y \dot{y})\, dt$$

$$= \int_0^T (-\dot{x}^2 - \dot{y}^2)\, dt < 0$$

unless $\dot{x} = \dot{y} = 0$ just a fixed pt

contradiction!

Ex 1 Does

$$\dot{x} = y + 2xy^2$$
$$\dot{y} = x + 2x^2y$$

have a closed orbit.

So we ask, is it gradient? If so

then

$$-V_x = y + 2xy^2$$
$$-V_y = x + 2x^2y$$

Cross-diff $\Rightarrow$

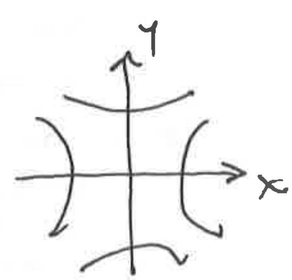
$$-V_{xy} = 1 + 4xy$$
$$-V_{yx} = 1 + 4xy$$

> same so yes

Here $-V = xy + x^2y^2$

So since it's gradient, no closed orbits

Furthermore. Consider $F = x^2 - y^2$

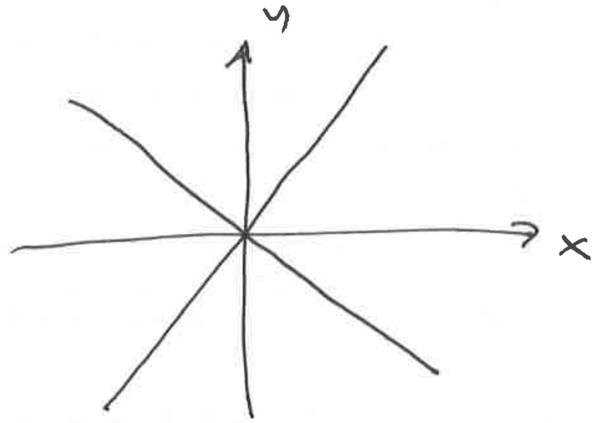


$$\dot{F} = 2x\dot{x} - 2y\dot{y}$$
$$= 2x(y + 2xy^2) - 2y(x + 2x^2y)$$
$$= 2xy + 4x^2y^2 - 2xy - 4x^2y^2 = 0$$

$\Rightarrow \quad x^2 - y^2 = c \quad \Leftarrow$ level curves!

If we include the origin, then

$$x^2 - y^2 = 0 \Rightarrow y = \pm x$$



we can also see this from the DE's

$$\frac{\dot{y}}{\dot{x}} = \frac{x + 2x^2y}{y + 2xy^2} = \frac{x(1+2xy)}{y(1+2xy)}$$

if $1+2xy = 0$ curve of crit pt.

if $1+2xy \neq 0$

$$\frac{dy}{dx} = \frac{x}{y} \Rightarrow y^2 = x^2 + c$$

and again, curves through $(0,0) \Rightarrow c = 0$

& $y = \pm x$

Bendixson's Negative Criterion

If R is a simply connected region in the phase plane and if

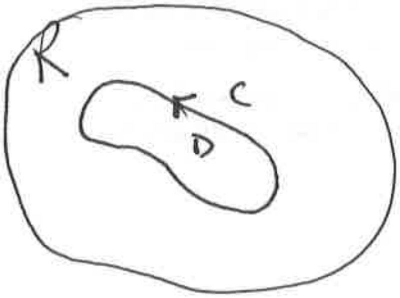
$$\dot{x} = f(x,y) \quad \dot{y} = g(x,y)$$

if $f_x + g_y$ is of one sign in R there are no closed orbits (paths) in R

Proof: By contradiction

Suppose there is a closed path in R and there $f_x + g_y > 0$ (<0 goes similarly)

Let D be the interior of the region of C the closed path. Then



$$\iint_D (f_x + g_y)\, dA > 0$$

Recall Green's Th^m in the plane $\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$

so $\displaystyle\iint_D (f_x + g_y)\,dA = \int_C -g\,dx + f\,dy = \int_C \underbrace{-g f\,dt + f g\,dt}_{\equiv 0}$

which leads to our condiction.

Now $\frac{dx}{dt} = f \quad \frac{dy}{dt} = g$

so $-g\,dx + f\,dy = -gf\,dt + fg\,dt \equiv 0$

$\Rightarrow \int_C 0 = 0$ leading to our contradiction

Ex 2 Show $\dot{x} = x + x^3 + y^2$

$\dot{y} = -x + y + x^2 y$

has no periodic orbits.

So here $f = x + x^3 + y^2$

$g = -x + y + x^2 y$

$$\left.\begin{array}{l} f_x = 1 + 3x^2 \\ g_y = 1 + x^2 \end{array}\right\} \text{ so } f_x + g_y = 2 + 4x^2 \geq 2$$

for all (x, y)

So by Bendixson's negative criterion

there are no periodic orbits.

Ex3 Consider

$$\dot{x} = y$$
$$\dot{y} = y + y^2 + x^2 y$$

Here $f = y$, $g = y + y^2 + x^2 y$

$f_x = 0 \quad g_y = 1 + 2y + x^2$

$f_x + g_y = 1 + 2y + x^2$ ← Not necessarily > 0 or < 0

So Ben. Neg. Crit ~~doesn't not~~ apply

Dulac (1933) improved Bend Neg Crit (1901)

Bendixson-Dulac Th^m

If $\beta(x, y)$ is cont^s diff on some region R

if $\frac{\partial}{\partial x}(\beta f) + \frac{\partial}{\partial y}(\beta g)$ is of 1 sign in R

then there are no periodic sol^n's.

Proof: Next HW.

so for our last ex.

$$(\beta y)_x + ((y+y^2+x^2y)\beta)_y = ?$$

Try $\beta = \beta(x)$ only

$$y\,\beta_x + \beta(1+2y+x^2)$$

$$y(\beta_x+2\beta) + \beta(1+x^2)$$

could we pick β such that

(i) $\beta_x + 2\beta = 0$

(ii) β is of 1 sign

i) $\beta = \beta_0 e^{-2x}$ & (ii) yes pick $\beta_0 = 1$

So by Bendixson-Dulac Th^m, there are no periodic orbits