VANDERBILT UNIVERSITY School of Engineering

Discrete Structures

CS 2212

(Fall 2020)

11 – Functions

Inverse of a Function

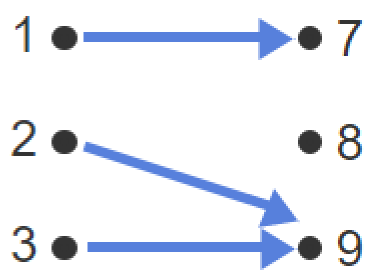
If a function $\boldsymbol{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a bijection, then the inverse of $\boldsymbol{f}$ is obtained by *exchanging* the first and second entries in each pair in $\boldsymbol{f}$.

What is the inverse of the following function?

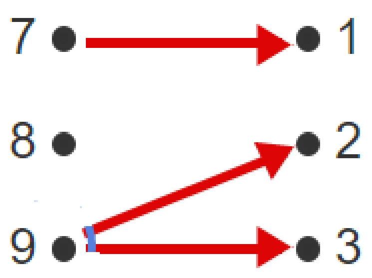
In fact, f^{1} is not a function.
The inverse of f **does not exist**.

X = { 1, 2, 3 }

Y = { 7, 8, 9 }

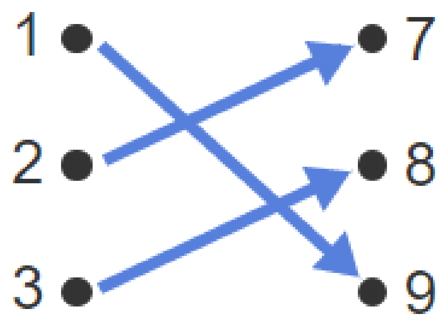


f: X → Y



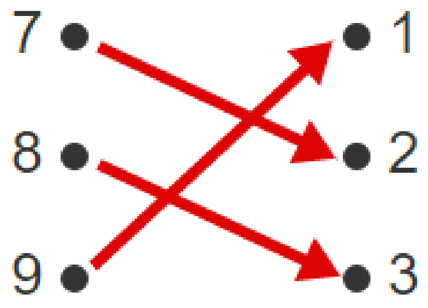
Inverse of a Function

What is the inverse of the following function?



$g: X \to Y$

$g = \{ (1, 9), (2, 7), (3, 8) \}$



$g^{-1} = Y \to X$

$g^{-1} = \{ (7, 2), (8, 3), (9, 1) \}$

g^{-1} is a function. g has an inverse defined by

$g^{-1}(7) = 2$

$g^{-1}(9) = 1$

$g^{-1}(8) = 3$

Lets look at an interesting application of

bijective functions

(Not in ZyBook)

Bijective Functions

Bijective functions are extremely useful. Lets see one of their application.

Problem: Lets say we have two different sets, and we want to see if they are of the **same cardinality** or not?

One way is to compute the cardinality of each individual set, and then see if they are same or not.

(Not a very general approach. What if we have sets of extremely large cardinality (may be infinity…)?)

Is there any other (more general) way without explicitly counting the number of elements in sets?

Bijective Functions and Set Cardinality

Set A
Group of people

Set B
No. of seats in a stadium

Without counting people or seats, how can we say that there are as many people as seats?

Simply ask people to take seats. If everybody has a seat and no seat is left, there are as many seats as the number of people.

In fact, all we are doing is trying to see if there is a ***bijective map*** between A and B.

Bijective Functions and Set Cardinality

Lets state it in general terms.

Consider two sets **A** **B**

To show they have the same cardinality, all we need to show is that there exists a bijective map between them.

Makes sense …

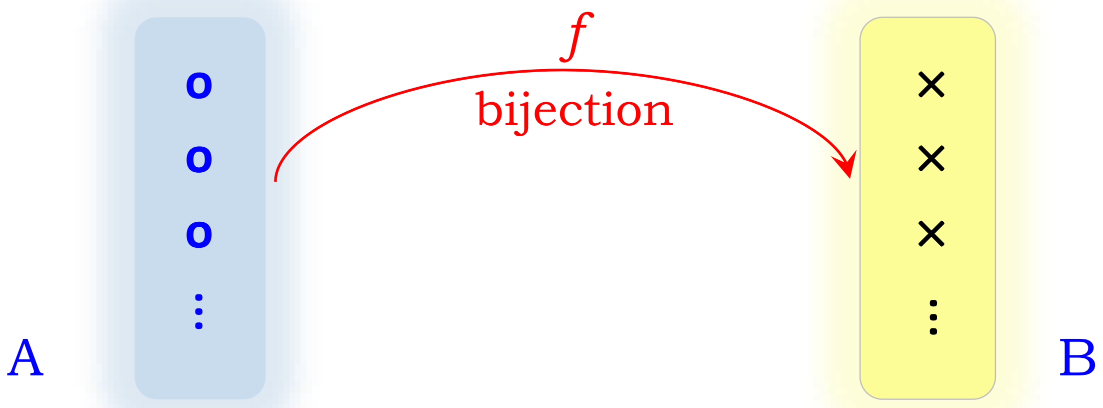
In fact, two sets have the same cardinality if and only if there is a bijective map between them

Bijective Functions and Set Cardinality

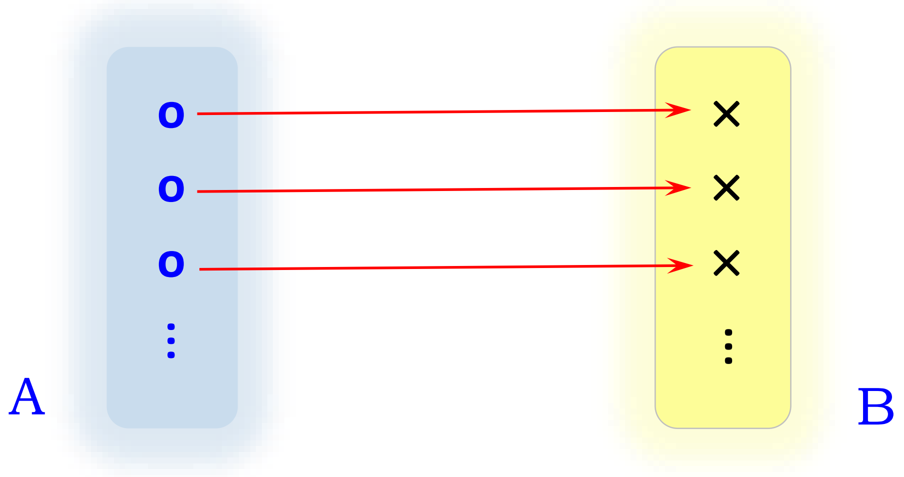
Problem: How to prove (without counting) that two sets **A** and **B** have the same cardinality?

Fact: If $f : A \to B$ a bijective function, then $|A| = |B|$

So, for the above problem, all we need to show is that there exists a bijective function from **A** to **B**.



Bijective Functions and Set Cardinality



(Injection) Every element in domain is mapped to a distinct element in co-domain.

(Surjection) Every element in co-domain is an image of some element in the domain.

$|A| = |B|$

1. There is a regular pattern mapping that can continue forever.
2. If we continue this mapping, we will never run out of numbers to pair up from each set
3. We will never skip any numbers in either set.

Bijective Functions and Set Cardinality

Question: Given $N^+ = \{1, 2, 3, \ldots\}$ and $W = \{0, 1, 2, 3, \ldots\}$, are these two sets the same size?

At first glance, we think probably not ….

Let's dig a little deeper the **bijective perspective**.

If we can establish that a bijective relationship exists between Z^+ and W, we would know $|N^+| = |W|$.

Lets try.

Bijective Functions and Set Cardinality

How about the following correspondence:

$1 \rightarrow 0$

$2 \rightarrow 1$

$3 \rightarrow 2$

$4 \rightarrow 3$ and so on.

Is this a one-to-one and onto correspondence?

How would we know?

Bijective Functions and Set Cardinality

To show something has an one-to-one bijective correspondence, we must show the following:

1. There is a regular pattern mapping that can continue forever.
2. If we continue this mapping, we will never run out of numbers to pair up from each set
3. We will never skip any numbers in either set.

Bijective Functions and Set Cardinality

Question 1: Is there a regular pattern here we can continue forever?

Answer:

Yep.

- Let n symbolizes an element from N^+ . We can always subtract one from n and map it to W. We can continue this pattern forever.

- We can also take an element w from W and add 1 to w and map it to N^+. We can continue doing this pattern forever.

Bijective Functions and Set Cardinality

Question 2: Will we ever run out of numbers to pair up in $N^+ = \{1, 2, 3, 4, \ldots\}$ and $W = \{0, 1, 2, 3, 4, \ldots\}$?

Answer: No.

Question 3: Will we ever skip any numbers?

Answer: No.

Conclusion: We have a bijective function between N^+ and W. So the infinite sets N^+ and W are of the same size.

Bijective Functions and Set Cardinality

Question: Given $N^+ = \{1, 2, 3, \ldots\}$ and $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, are these two sets the same size?

How about the following correspondence:

$1 \rightarrow 0$

$2 \rightarrow -1$

$3 \rightarrow 1$

$4 \rightarrow -2$

$5 \rightarrow 2$ and so on.

Odd numbers (> 1): $(n-1)/2$
Even numbers: $-n/2$

Thus, it must be the case that

|N+| = |Z|

Bijective Functions and Set Cardinality

Some important points to remember:

- There can be multiple bijective functions
- Sometimes, it is tricky to find a bijective function (as we will see in our next example).
- If we cannot find a bijective map, we cannot conclude that sets are not of the same cardinality.
- In fact, to make the conclusion [*sets do not have the same cardinality*], we must prove that there does not exist a bijective function between those two sets.

Bijective Functions and Set Cardinality

Lets see another very interesting result.

There are as many positive rational numbers as positive integers.

Seems counter intuitive …

Approach: All we need is a bijective function f

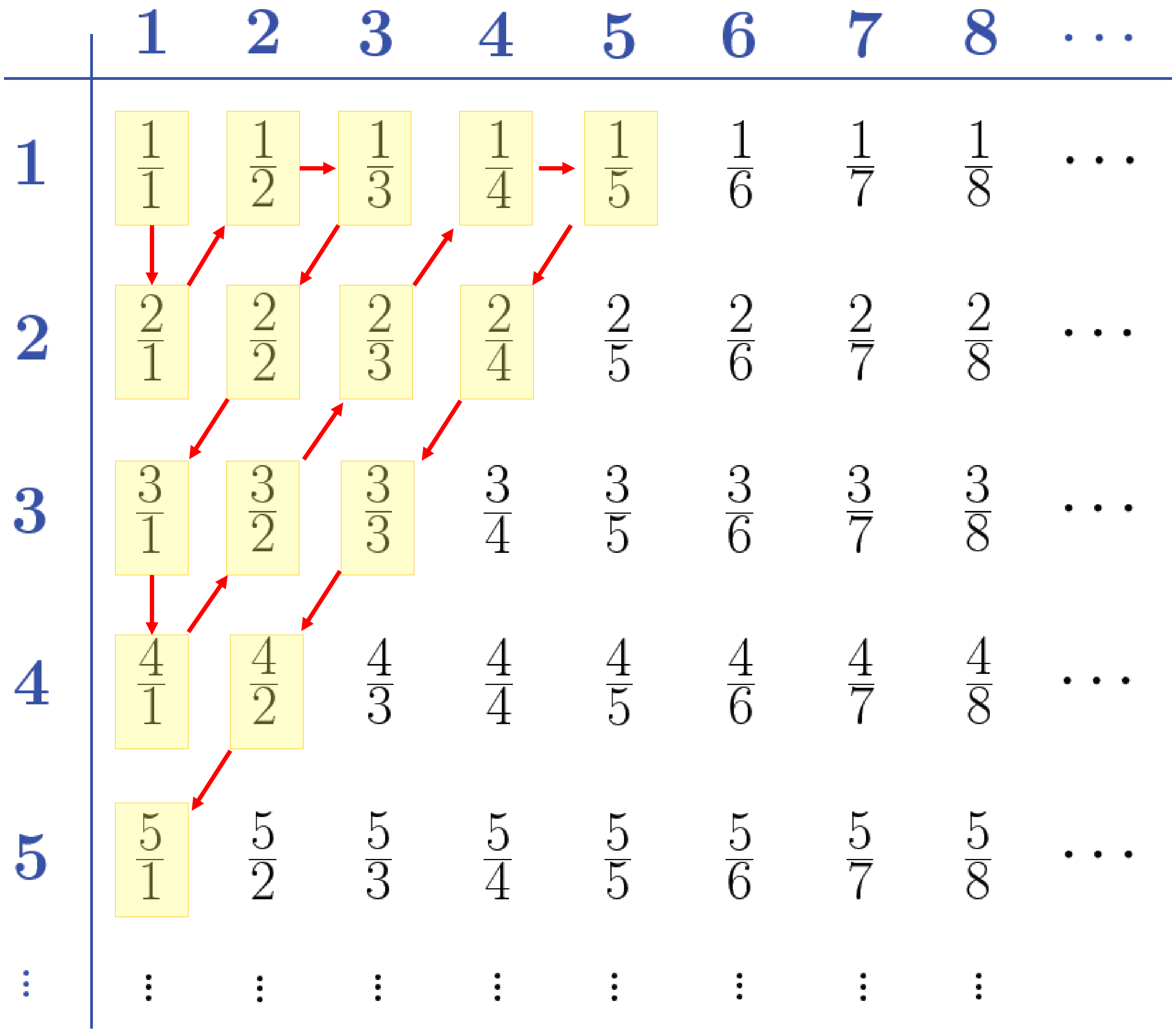
$$\boldsymbol{f : \mathbf{Q}^+ \rightarrow \mathbf{Z}^+}$$

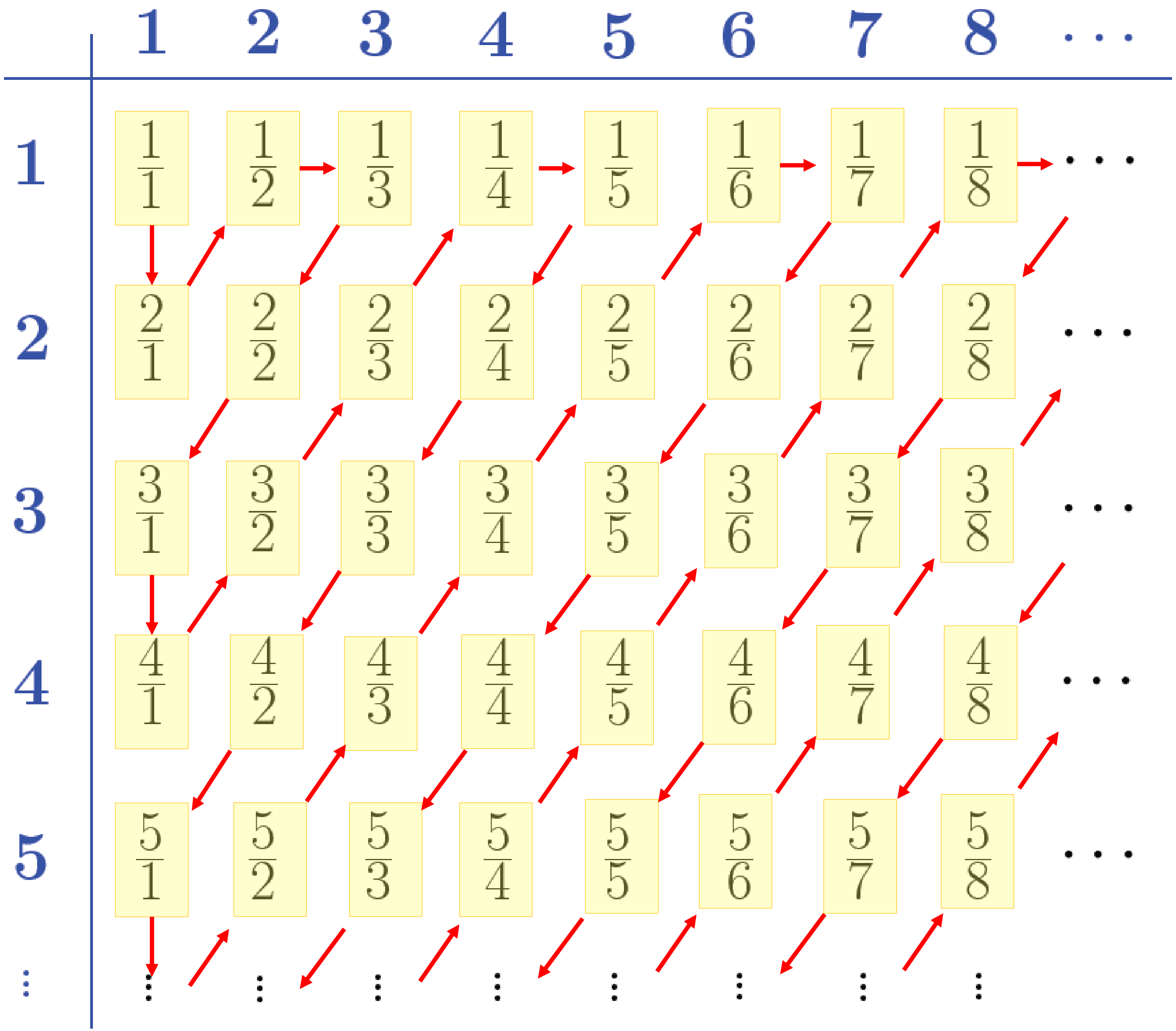
Lets try to find such a function using a very clever **"diagonal argument"**.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	$\cdots$
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\cdots$
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\frac{2}{7}$	$\frac{2}{8}$	$\cdots$
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	$\cdots$
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	$\cdots$
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

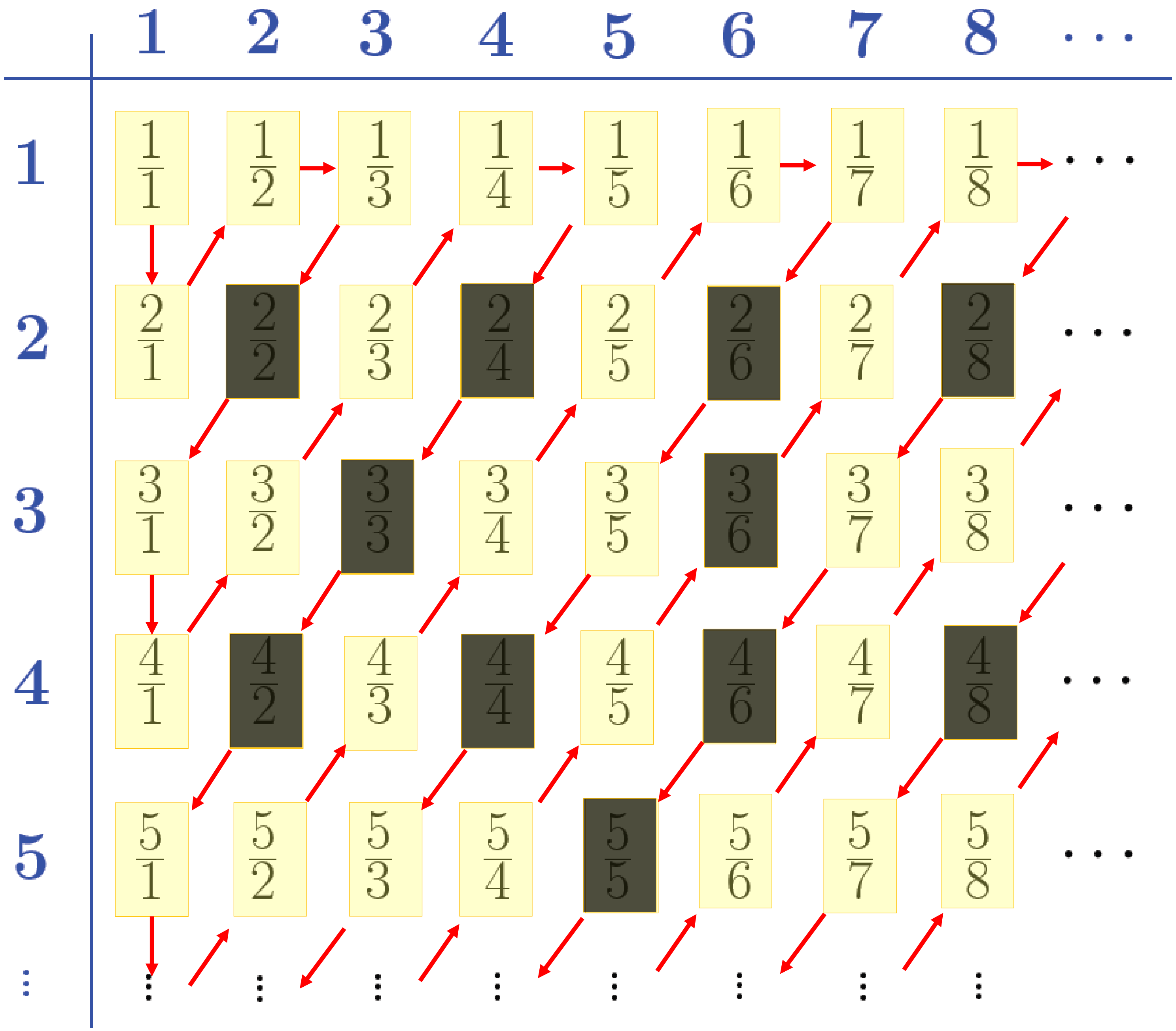
	1	2	3	4	5	6	7	8	$\cdots$
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\cdots$
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\frac{2}{7}$	$\frac{2}{8}$	$\cdots$
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	$\cdots$
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	$\cdots$
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

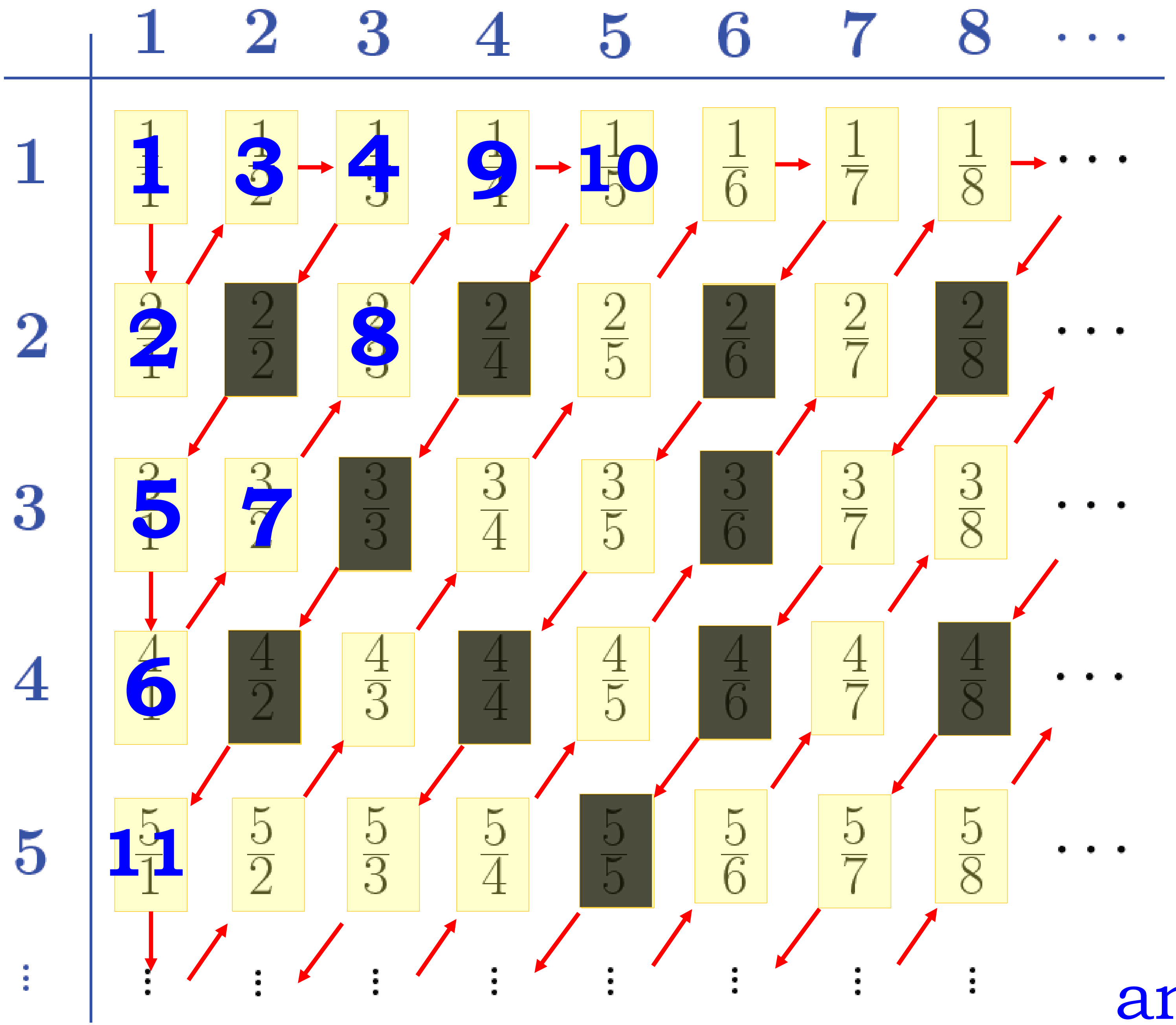
Can we **"list"** all these rational numbers in this infinite two dimensional grid?


1 2 3 4 5 6 7 8 ...
1: 1/1 1/2 1/3 1/4 1/5 1/6 1/7 1/8 ...
2: 2/1 2/2 2/3 2/4 2/5 2/6 2/7 2/8 ...
3: 3/1 3/2 3/3 3/4 3/5 3/6 3/7 3/8 ...
4: 4/1 4/2 4/3 4/4 4/5 4/6 4/7 4/8 ...
5: 5/1 5/2 5/3 5/4 5/5 5/6 5/7 5/8 ...



1 2 3 4 5 6 7 8 …
1 1/1 1/2 1/3 1/4 1/5 1/6 1/7 1/8 …
2 2/1 2/2 2/3 2/4 2/5 2/6 2/7 2/8 …
3 3/1 3/2 3/3 3/4 3/5 3/6 3/7 3/8 …
4 4/1 4/2 4/3 4/4 4/5 4/6 4/7 4/8 …
5 5/1 5/2 5/3 5/4 5/5 5/6 5/7 5/8 …
⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮


1 2 3 4 5 6 7 8 ...
1 1/1 1/2 1/3 1/4 1/5 1/6 1/7 1/8 ...
2 2/1 2/2 2/3 2/4 2/5 2/6 2/7 2/8 ...
3 3/1 3/2 3/3 3/4 3/5 3/6 3/7 3/8 ...
4 4/1 4/2 4/3 4/4 4/5 4/6 4/7 4/8 ...
5 5/1 5/2 5/3 5/4 5/5 5/6 5/7 5/8 ...
⋮


and so on ...

Bijective Functions and Set Cardinality

Thus, we can list all positive rational numbers.

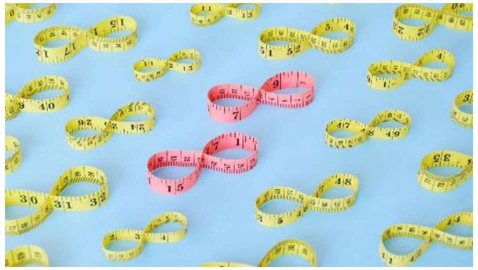
Simply, for every positive rational number, we have exactly one positive integer using this bijection.

$$\boldsymbol{f} : \mathbf{Q}^+ \rightarrow \mathbf{Z}^+$$

Consequently, there are as many positive rational numbers as positive integers

Bijective Functions and Set Cardinality

Question: Are all infinite sets really the same size?



No

For instance, the set of **real** numbers is not the same size as the set of **integers**.

Why?

George Cantor (in 1870's) proved that there does not exist a bijective map between them.

Countable and Uncountable Sets

A set C is **countably infinite** if there exists

$$\boldsymbol{f : C \to \{1, 2, 3, \ldots\}},$$

where, f is a **bijection**.

Examples:

N, Z, Q, pairs of N, triplets of N, …

Finite sets are always countable.

A set that is not countable is called **uncountable set**.

A nice blog read for your free time: "Does Cantor's Diagonalization Proof Cheat?"
(https://rjlipton.wordpress.com/2010/06/11/does-cantors-diagonalization-proof-cheat/)

Functions - Log

Here are some important properties of the log function:

$$\log_b 1 = 0$$

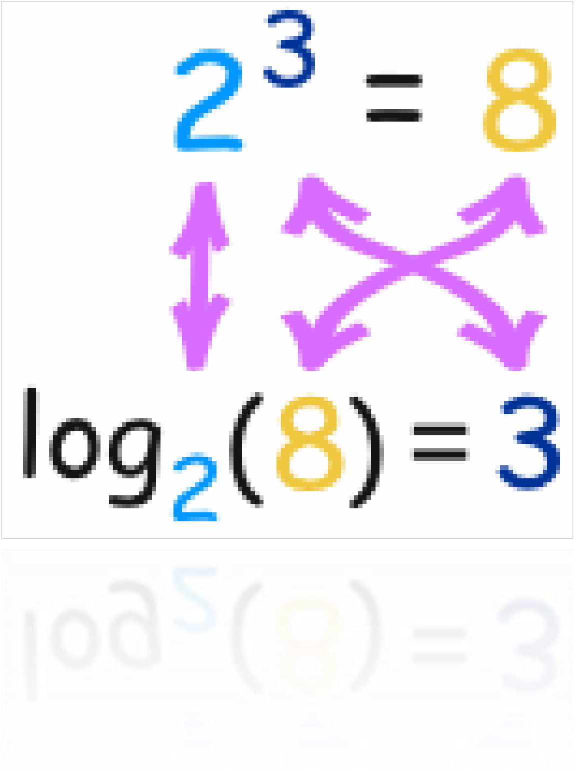
$$\log_b b = 1$$

$$\log_b b^x = x$$

$$\log_b (xy) = \log_b x + \log_b y$$

$$\log_b x^y = y \log_b x$$

$$\log_a x = (\log_a b)(\log_b x)$$



Functions - Log

Practice: Estimate the value of $\log_2(5^2 2^5)$

Approach: We can get pretty close by finding its upper and lower bounds.

Answer:

$$\log_2(5^2 2^5) = \log_2(5^2) + \log_2(2^5) = \log_2(5^2) + 5$$

We know that $16 < 5^2 < 32$

So this implies that $4 + 5 < \log_2(5^2 2^5) < 5 + 5$

Therefore $9 < \log_2(5^2 2^5) < 10$

Functions - Exponentiation

- There's a section in zyBooks on **exponentiation** in *Chapter 4*. If you've not done math with exponents before, you should read it over.
- In terms of computer science, we often refer to log notation and exponentiation when analyzing the performance of an algorithm.
- We'll be revisiting this topic later in *Chapter 6* when we discuss algorithm analysis.