Math 4381|6378 Sym

Invariance of 1st order ODEs

Given $$\frac{dy}{dx} = F(x,y)$$

If this is invariant under the infinitesimal transformations

$$\bar{x} = x + \varepsilon X(x,y) + O(\varepsilon^2)$$
$$\bar{y} = y + \varepsilon Y(x,y) + O(\varepsilon^2)$$

then X, Y & F satisfy Lie's invariance cond.

$$Y_x + (Y_y - X_x)F - X_y F^2 = X F_x + Y F_y$$

Any nonzero X & Y (both - singly ok) will work.

Solving $\quad X r_x + Y r_y = 0, \quad X S_x + Y S_y = 1$

will give a C of V leading to a separable ODE!

We now want to extend to

- higher order ODE
- systems ODE
- PDEs

etc

So now we look for structure.

1st let's consider a particular exampl.

$$\bar{x} = x + \varepsilon x^2 + O(\varepsilon^2)$$

$$\bar{y} = y + \varepsilon xy + O(\varepsilon^2)$$

let's consider $\dfrac{d\bar{y}}{d\bar{x}}$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d}{dx}(y + \varepsilon xy)}{\frac{d}{dx}(x + \varepsilon x^2)} = \frac{y' + \varepsilon(xy' + y)}{1 + 2\varepsilon x} + O(\varepsilon^2)$$

$$= \{y' + \varepsilon(xy' + y)\}\{1 - 2\varepsilon x\} + O(\varepsilon^2)$$

$$= y' + \varepsilon(y - xy') + O(\varepsilon^2)$$

so in general

if $\quad \bar{x} = x + \varepsilon X(x,y) + O(\varepsilon^2)$

$$\bar{y} = y + \varepsilon Y(x,y) + O(\varepsilon^2)$$

then $\quad \bar{y}' = y' + \varepsilon Y_{[x]}(x,y,y') + O(\varepsilon^2)$

so $\quad Y_{[x]}$ will contain y'

so $\quad \bar{y}'' = y'' + \varepsilon Y_{[xx]}(x,y,y',y'') \cdots + O(\varepsilon^2)$

$$\bar{y}^{(n)} = y^{(n)} + \varepsilon Y_{[x(n)]}(x,y,y', \ldots y^{(n)}) + O(\varepsilon^2)$$

Let's calculate derivatives in general

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d}{dx}(y + \varepsilon Y)}{\frac{d}{dx}(x + \varepsilon X)} + O(\varepsilon^2)$$

Introduce $\quad D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots$

total differential operator

$$\frac{d\bar{y}}{d\bar{x}} = \frac{D_x(y+\varepsilon Y)}{D_x(x+\varepsilon X)} + O(\varepsilon^2)$$

$$= \frac{y' + \varepsilon D_x(Y)}{1+\varepsilon D_x(X)}$$

$$= \{y' + \varepsilon D_x(Y)\}\{1-\varepsilon D_x(X)\}$$

$$= y' + \varepsilon\{D_x(Y) - y' D_x(X)\} + O(\varepsilon^2)$$

So $\quad Y_{[x]} = D_x(Y) - y' D_x(X)$

Similarly

$$\frac{d^2\bar{y}}{d\bar{x}^2} = \frac{\frac{d}{dx}(\bar{y}')}{\frac{d}{dx}\bar{x}} = \frac{D_x(y' + \varepsilon Y_{[x]})}{D_x(x+\varepsilon X)}$$

$$= \frac{y'' + \varepsilon D_x(Y_{[x]})}{1+\varepsilon D_x(X)} + O(\varepsilon^2)$$

$$= \{y'' + D_x(Y_{[x]})\varepsilon\}\{1 + D_x(X)\varepsilon\} + O(\varepsilon^2)$$

$$\frac{\delta^2 \bar{y}}{\delta \bar{x}^2} = y'' + \varepsilon \left\{ D_x(Y_{[x]}) - y'' D_x(x) \right\} + O(\varepsilon^2)$$

so $Y_{[xx]} = D_x(Y_{[x]}) - y'' D_x(x)$

so for

$$Y_{[x]} = D_x(Y) - y' D_x(x)$$

$$Y_{[xx]} = D_x(Y_{[x]}) - y'' D_x(x)$$

$$Y_{[xxx]} = D_x(Y_{[xx]}) - y''' D_x(x)$$

so we can define these recursively

Let's return to Lie's inv. condition

$$\underbrace{Y_x + (Y_y - X_x)F - X_y F^2}_{} = X F_x + Y F_y$$

this is $Y_{[x]}\Big|_{y'=F}$

so $\quad Y[x] - X F_x - Y F_y \Big|_{y'=p} = 0$

our ODE is $\quad \Delta = y' - F(x,y)$

if we introduce the infinitesimal operator

$$\Gamma = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y}$$

then $\quad X F_x + Y F_y = \Gamma(F)$

we introduce the extended operator

$$\Gamma^{(1)} = \Gamma^{t} + Y[x] \frac{\partial}{\partial y'}$$

then Lie's invariance condition is

$$\Gamma^{(1)} \Delta \Big|_{\Delta=0} = 0$$

← real simple looking

So how would this extend to 2nd order ODE?

if $y'' = F(x, y, y')$

let $\Delta = y'' - F(x, y, y')$

let $\Gamma = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y}$

$$\Gamma^{(1)} = \Gamma + Y[x] \frac{\partial}{\partial y'}$$

$$\Gamma^{(2)} = \Gamma^{(1)} + Y(xx) \frac{\partial}{\partial y''}$$

Lie's inv. conditi

$$\Gamma^{(2)} \Delta \Big|_{\Delta = 0} = 0$$

Let us start 1 example

$$y'' = 0$$

so $\Delta = y''$

So $\Gamma^{(2)} \Delta = 0$

$\Rightarrow \quad Y_{[xx]} = 0$ Still need to apply $y'' = 0$

Let's get a formula for $Y_{[xx]}$

Now $$Y_{[x]} = \underbrace{Y_x + Y_y y'}_{D_x(Y)} - y' \underbrace{(X_x + X_y y')}_{D_x(X)}$$

$$Y_{[xx]} = D_x\left(Y_x + Y_y y' - y'(X_x + X_y y')\right) - y''(X_x + X_y y')$$

$$= Y_{xx} + Y_{xy} y'$$
$$+ Y_{xy} y' + Y_{yy} y'^2 + Y_y y''$$
$$- y''(X_x + X_y y')$$
$$- y'\left(X_{xx} + X_{xy} y' + X_{xy} y' + X_{yy} y'^2 + X_y y'' \right)$$
$$- y''(X_x + X_y y')$$

$$Y_{[xx]} = Y_{xx} + (2Y_{xy} - X_{xx})y' + (Y_{yy} - 2X_{xy})y'^2 - X_{yy}y'^3$$

$$+ (Y_y - 2X_x)y'' - 3X_y y' y''$$

I bet these would be easy to generate with maple

so $Y_{[xx]}\Big|_{y''=0}$

$$\Rightarrow \quad Y_{xx} + (2Y_{xy} - X_{xx})y' + (Y_{yy} - 2X_{xy})y'^2 - X_{yy}y'^3 = 0$$

$\because$ X, Y independent of y'

then

$$\left.\begin{aligned} Y_{xx} &= 0 \\ 2Y_{xy} - X_{xx} &= 0 \\ Y_{yy} - 2X_{xy} &= 0 \\ X_{yy} &= 0 \end{aligned}\right\}$$

an over determined system for X & Y.

so $X_{yy} = 0$

$\Rightarrow$ $X = a(x)y + b(x)$ $\quad$ a, b arbitrary

$Y_{yy} = 2X_{xy}$
$= 2a'(x)$

$Y_y = 2a'(x)y + p(x)$

$Y = a'(x)y^2 + p(x)y + q(x)$ $\quad$ p, q arb

Now sub into remaining 2 eq^n's

$Y_{xx} = 0$

$a'''(x)y^2 + p''(x)y + q''(x) = 0$

$\Rightarrow$ $a''' = 0$
$p'' = 0$
$q'' = 0$

$2Y_{xy} - X_{xx} = 0 \Rightarrow 4a''(x)y + 2p'(x) - a''(x)y - b''(x) = 0$

$\Rightarrow$ $3a''(x) = 0, \quad 2p'(x) - b''(x) = 0$

So $a'' = 0 \Rightarrow a = c_1 x + c_2$

$p'' = 0 \Rightarrow p = c_3 x + c_4$

$q'' = 0 \Rightarrow q = c_5 x + c_6$

Now $b'' = 2p' = 2c_3$

$b' = 2c_3 x + c_7$

$b = c_3 x^2 + c_7 x + c_8$

So $X = (c_1 x + c_2) y + c_3 x^2 + c_7 x + c_8$

$Y = c_1 y^2 + (c_3 x + c_4) y + c_5 x + c_6$

We need more examples

and Maple!