Calculus Review 1

Random Topics include limits/asymptotes, max/min, implicit differentiation, related rates, instantaneous rate of change, graphing, integrals, increasing/decreasing intervals, and more.

Limits and Asymptotes Puzzle

Use the limits and asymptotes "clues" to figure out the function!

A certain rational function $f(x)$ has quadratic functions in both its numerator and denominator.

Also, it has these characteristics:

$f(x)$ has a vertical asymptote at $x = 5$

$f(x)$ has one x-intercept, at $x = 3$

$f(x)$ is (removably) discontinuous at $x = 1$, with $\lim_{x\to 1} f(x) = \frac{-1}{7}$

a) What is the function $f(x)$?

b) $f(0) =$

c) Sketch a graph of the function.

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a) Using the "clues",

vertical asymptote at $x = 5$ $\quad f(x) = \frac{\quad}{(x-5)}$

x-intercept at $x = 3$ $\quad f(x) = \frac{(x-3)}{(x-5)}$

removable discontinuity at $x = 1$ ("hole") $\quad f(x) = \frac{(x-3)(x-1)}{(x-5)(x-1)}$

**The shape of the function is determined by the "a" value $\quad f(x) = \frac{a\,(x-3)(x-1)}{(x-5)(x-1)}$

To find the a value, we need another point, or use the limit!

$$\frac{a(1-3)}{(1-5)} = \frac{-1}{7} \qquad \frac{-2a}{-4} = \frac{-1}{7} \qquad -14a = 4 \qquad a = \frac{-2}{7}$$

$$f(x) = \frac{\frac{-2}{7}(x-3)(x-1)}{(x-5)(x-1)} \quad \text{or,} \quad \frac{-2x^2 + 8x - 6}{7x^2 - 42x + 35}$$

b) $f(0)$ is the y-intercept... $\frac{-6}{35}$

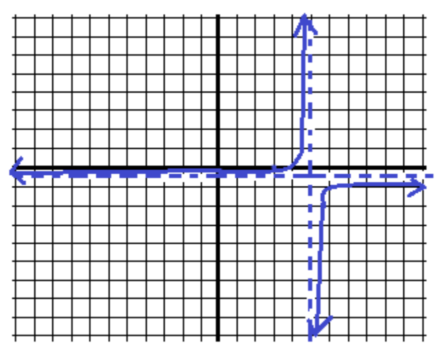
c) To sketch, we'll utilize the intercepts, asymptote, and limits....

horizontal asymptote: $y = -2/7$

vertical asymptote: $x = 5$

x-intercept: (3, 0)

x	f(x)
-3	-3/14
0	-6/35
3	0
4	2/7
6	-6/7
7	-4/7



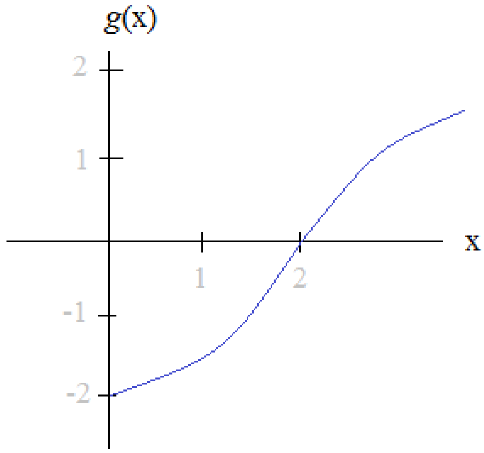
(sketch)

Determine if the value is < > or = to zero:

a) $g(1)$

b) $g'(1)$

c) $g''(1)$



Answers and explanations:

a) For any function,
if the output is positive, it is above the x-axis
if the output is negative, it is below the x-axis
and, if the output is 0, it is on the x-axis....
Since the output of g(1) is below the x-axis, it is negative.
< 0

"position"

b) The first derivative is the function's instantaneous rate of change (slope).
If the slope is positive (upward), then the value is positive.
If the slope is negative (downward), then the derivative is negative.
and, if the slope is 0 (horizontal), then the derivative equals 0.
Since the output of g(1) is on an upward sloping part of the curve,
the first derivative g'(1) is > 0

"slope"

c) The second derivative represents the 'acceleration' or rate the slope is changing..
If the slope is increasing, the curve is concave up and the second derivative is positive.
If the slope is decreasing, the curve is concave down and the second derivative is negative.
If it is at a 'point of inflection', the curve is neither concave up nor down, so, the second derivative is zero.

Since the output of g(1) is on a part that is concave up, the second derivative g''(1) must be > 0

"concavity"

Example: Two particles that move along a horizontal axis have the following models:

$$x(t) = 3\cos\left(\frac{\pi}{4}t\right) \qquad s(t) = t^3 - 6t^2 + 9t + 4$$

On the interval $0 \le t \le 6$, when do the particles move in the same direction?

Find the intervals where each particle increases and decreases...

First derivative.... $x'(t) = -3\sin\left(\frac{\pi}{4}t\right) \cdot \frac{\pi}{4}$ $s'(t) = 3t^2 - 12t + 9 + 0$

Then, set equal to zero (to find where particle changes direction)

$$-3\frac{\pi}{4}\sin\left(\frac{\pi}{4}t\right) = 0$$

$$\sin\left(\frac{\pi}{4}t\right) = 0$$

$t = 4k$ (where k is any integer)

So, in interval [0, 6], 0 and 4

$$3t^2 - 12t + 9 = 0$$

$$3(t^2 - 4t + 3) = 0$$

$$(t-3)(t-1) = 0$$

$t = 1$ and 3

Then, test each sub-interval to determine whether increasing or decreasing...

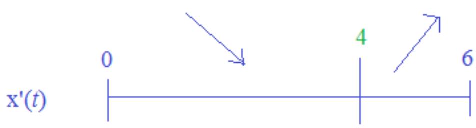
$$x'(1) = -3\frac{\pi}{4}\sin\left(\frac{\pi}{4}1\right) < 0$$

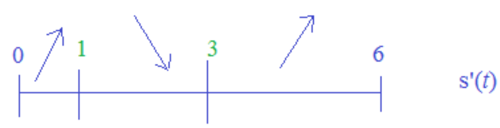
$$x'(5) = -3\frac{\pi}{4}\sin\left(\frac{\pi}{4}5\right) > 0$$

$$s'(1/2) = 3(1/2 - 3)(1/2 - 1) > 0$$

$$s'(2) = 2(2-3)(2-1) < 0$$

$$s'(4) = 2(4-3)(4-1) > 0$$

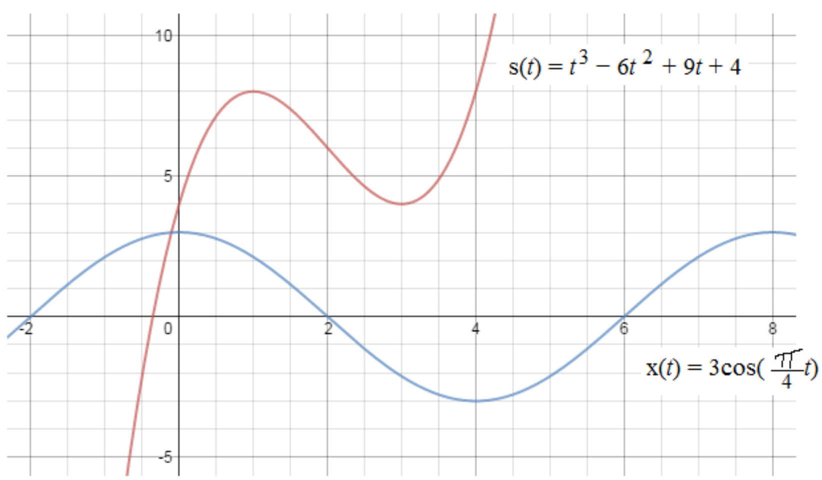




Finally, determine the sub-intervals where $x(t)$ and $s(t)$ move in the same direction....

Interval (1, 3) where both are decreasing (i.e. moving to the left)

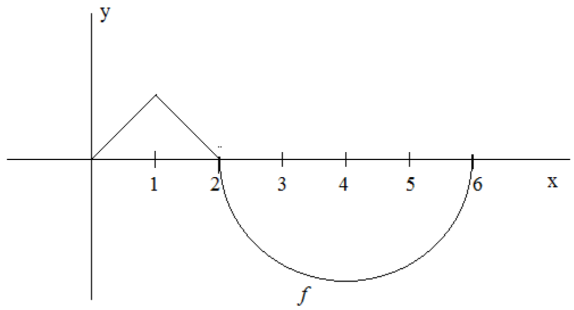
and, Interval (4, 6] where both are increasing (i.e. moving to the right)



Calculus Review Question

The graph of f consists of two line segments and a semicircle.

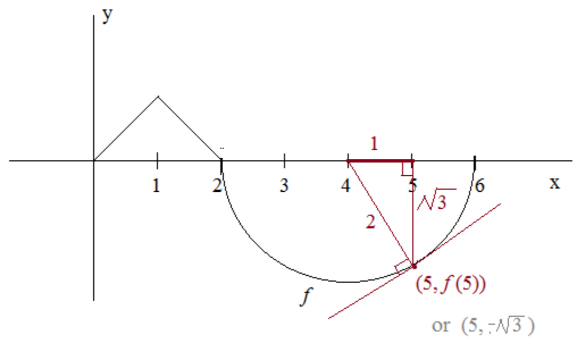
Find $f'(5)$



Solution 1: Recognizing that $f'(x)$ is the slope at a given point x, find the slope of a line tangent to the semicircle at x = 5

Since the span of the semicircle is 4 units, we know the radius would be 2 units.

Since the distance from 4 to 5 is one unit and we have constructed a right triangle, the vertical line segment length is $\sqrt{3}$



$$\text{Slope of radius} = \frac{\text{"rise"}}{\text{"run"}} = \frac{\Delta y}{\Delta x} = \frac{-\sqrt{3}}{1}$$

$$\text{Slope of perpendicular line} = \frac{1}{\sqrt{3}}$$

$$\boxed{f'(5) = \frac{1}{\sqrt{3}}}$$

(geometry note: a tangent line and the radius that shares the common point are perpendicular)

Solution 2: Find the derivative of f, and plug in 5

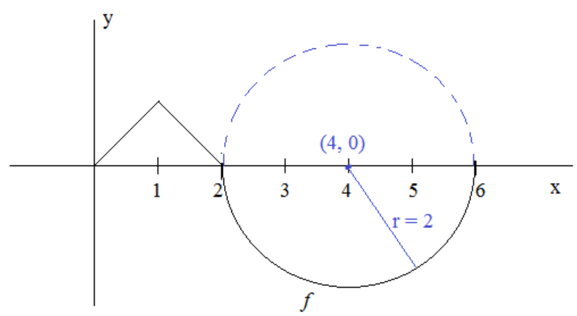
Since x = 5 involves the semicircle, ignore the line segments and describe the equation of the semicircle.

Standard form of a circle: $(x-h)^2 + (y-k)^2 = r^2$

where the radius = r
center = (h, k)

$(x-4)^2 + y^2 = 4$ where $y \leq 0$

(because it's a semicircle)



Find derivative of $(x-4)^2 + y^2 = 4$

a) $(x-4)^2 + y^2 = 4$ (solve for y)

$$y^2 = 4 - (x-4)^2$$

$$y = \pm\sqrt{4-(x-4)^2}$$

$$y = -\sqrt{4-(x-4)^2}$$

(since the range is ≤ 0, eliminate the positive solutions)

(find dy/dx)

$$\frac{dy}{dx} = -\frac{1}{2}\left(4-(x-4)^2\right)^{\frac{-1}{2}}\left(-2(x-4)\right)$$

$$= \frac{(x-4)}{\left(4-(x-4)^2\right)^{\frac{1}{2}}}$$

$$f'(5) = \frac{(5-4)}{\left(4-(5-4)^2\right)^{\frac{1}{2}}} = \boxed{\frac{1}{\sqrt{3}}}$$

b) Implicit differentiation

$$(x-4)^2 + y^2 = 4$$

$$2(x-4) + 2yy' = 0$$

$$2yy' = -2(x-4)$$

$$y' = \frac{-(x-4)}{y}$$

plug in the point $(5, -\sqrt{3})$

$$f'(5) = \frac{-(5-4)}{-\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}}}$$

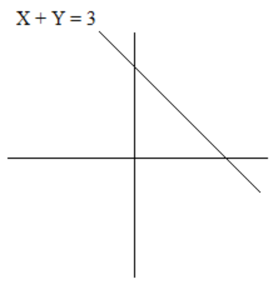
Calculus Review Question

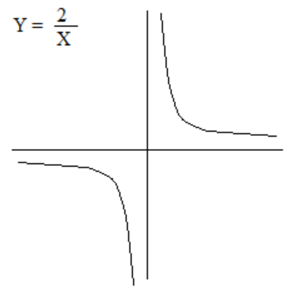
Find the area between the following equations: $X + Y = 3$

$Y = \frac{2}{X}$ And, sketch a graph.

$X + Y = 3 \longrightarrow Y = 3 - X$

$Y = \frac{2}{X}$





(Find where the equations intersect)

$3 - X = \frac{2}{X}$ (multiply by X)

$3X - X^2 = 2$ (factor and solve)

$X^2 - 3X + 2 = 0$

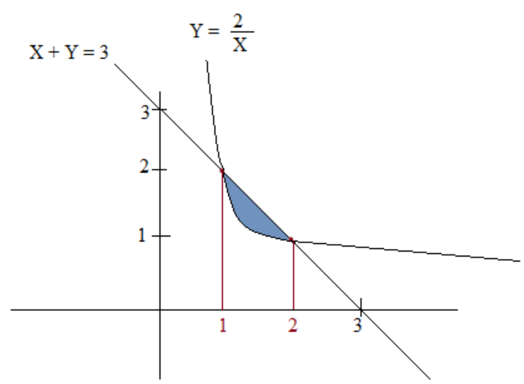
$(X - 1)(X - 2) = 0$

$X = 1, 2$

For $X = 1$, $(1) + Y = 3$, $Y = 2$ (plug X values into original equations to get Y values)

For $X = 2$, $(2) + Y = 3$, $Y = 1$

Equations intersect at (1, 2) and (2, 1)



Set up the definite integral to determine the area:

$$\int_1^2 (3 - X)\,dx - \int_1^2 \frac{2}{X}\,dx$$

(area under the line) (area under the curve)

$$3X - \frac{X^2}{2}\Bigg|_1^2 - 2\ln X\Bigg|_1^2$$

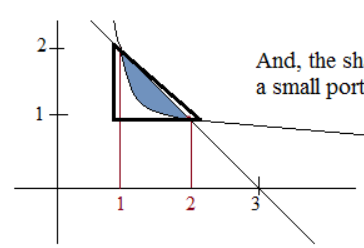
$$3(2) - \frac{(2)^2}{2} - \left(3(1) - \frac{(1)^2}{2}\right) - \Big[2\ln(2) - 2\ln(1)\Big]$$

$$6 - 2 - (3 - \frac{1}{2}) - 2\ln 2 + 0$$ $\ln(1) = 0$

$$\boxed{\frac{3}{2} - \ln 4}$$ $2\ln 2 = \ln(2)^2$

$$\boxed{1.5 - 1.386 = .114}$$

Check if reasonable:



The area of the triangle is 1/2 or .5

And, the shaded blue area is a small portion of that triangle.

Find the point on the graph $y = 3x^2$ on [1, 2] at which the tangent to the graph has *the same slope* as the line that passes through the endpoints of the closed interval.

Note: this problem is an application/verification of the mean value theorem!

Answer:

Step 1: Find the slope of the line that passes through the endpoints.

at $x = 1$, $y = 3$
and,
at $x = 2$, $y = 12$

slope between (1, 3) and (2, 12) is 9

Step 2: Find the point on the curve where the slope is 9

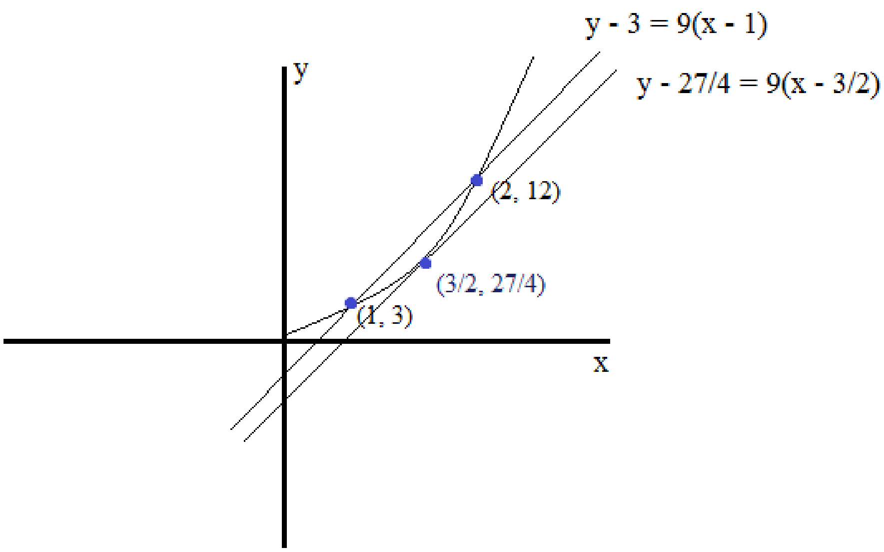
$y' = 6x$

$6x = 9$ when $x = 3/2$

at $x = 3/2$, $y = 3(3/2)^2 = 27/4$

The instantaneous rate of change/slope at (3/2, 27/4) = 9

Step 3: graph equations to verify



Find the $\lim_{x \to 0} \frac{\sin 2x}{x}$

Solutions:

method 1: trig identities

$\sin 2x = 2\sin x\cos x \qquad \lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\lim_{x \to 0} \frac{\sin 2x}{x} \longrightarrow \lim_{x \to 0} \frac{2\sin x\cos x}{x} = \lim_{x \to 0} \frac{2\cos x \cdot \sin x}{x}$$

$$\lim_{x \to 0} 2\cos x \cdot \lim_{x \to 0} \frac{\sin x}{x}$$

$$= 2 \times 1 = 2$$

method 2: calculus (l'hospital's rule)

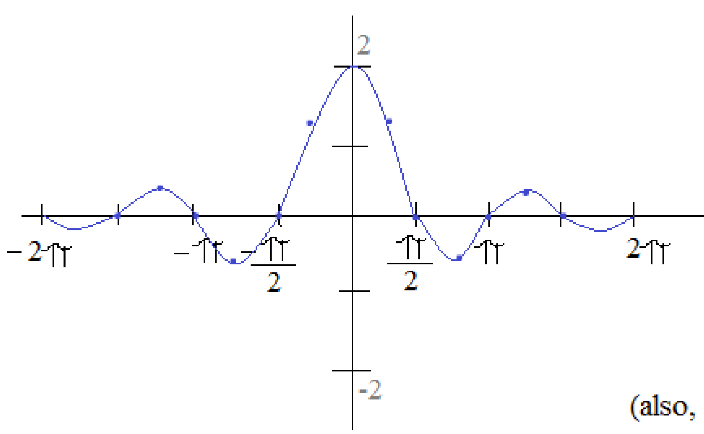
derivative of $\sin 2x = 2\cos 2x$
derivative of $x = 1$

$\lim_{x \to 0} \frac{\sin 2x}{x}$ using substitution: $\frac{0}{0}$ (since $\frac{0}{0}$, can use l'hospital's rule)

(after taking the 1st derivative) $\lim_{x \to 0} \frac{2\cos 2x}{1} = 2$

method 3: sketch graph and determine behavior

x	y
$-\pi$	0
$-3\pi/4$	$-.424$
$-\pi/2$	0
$-\pi/4$	1.27
0	?
.78	1.27
1.57	0
2.35	$-.424$
3.14	0
3.92	.255
4.71	0



(also, $\lim_{\pm\infty} \frac{\sin(2x)}{x} = 0$)

X	0	1	2	3
f	0	2	0	−2
f'	3	0	DNE	−3
f''	0	−1	DNE	0

X	$0 < X < 1$	$1 < X < 2$	$2 < X < 3$
f	+	+	−
f'	+	−	−
f''	−	−	−

The charts represent the function $f(X)$ on the interval (0, 3)

a) What are the absolute extrema?

b) What are the point(s) of inflection?

c) Sketch the graph of $f(X)$

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a) The function increases from 0 to 1, then it decreases from 1 to 3. (and, $f' = 0$ at x = 1). Therefore, the absolute maximum in the interval [0, 3] occurs at x = 1 (the coordinate (1, 2))

And, the minimum will occur at either x = 0 or x = 3... Since $f(0) = 0$ and $f(3) = -2$, the absolute minimum occurs at x = 3 (the coordinate (3, -2))

b) A point of inflection occurs when the second derivative equals zero.

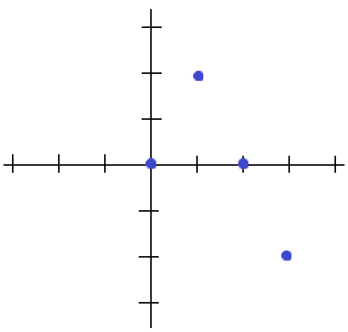
On the interval (0, 3), there are no points of inflection.

If the domain of the function were extended, there would be points of inflection at x = 0 and x = 3

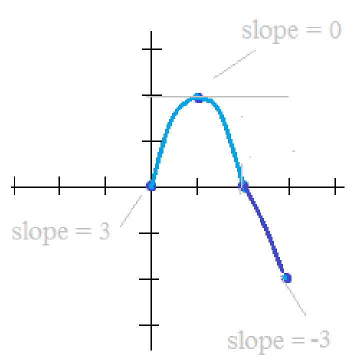
c) to sketch the graph, start with the function:

Coordinates will include

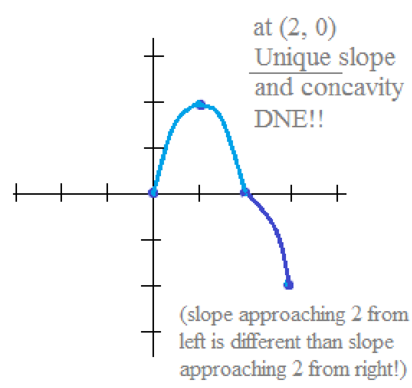
(0, 0) (1, 2) (2, 0) (3, -2)



then, use the first derivative f' to identify the instantaneous slope...



Use the 2 charts and second derivatives to smooth the curves....



Steps and Strategies for Calculus Related Rates of Change

Example:

The length of a rectangle is decreasing at a rate of 2 feet/minute.
The width of a rectangle is increasing at a rate of 2 feet/minute.

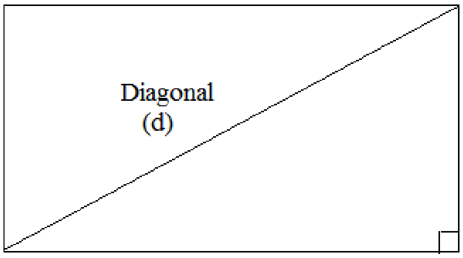
If the length is 12 feet and the width is 5 feet find the rates of the change of the:

a) Area

b) Perimeter

c) Diagonal Length

Step 1: Draw a picture and label given values



width (w) = 5

2 feet/minute (increasing)

length (l) = 12

2 feet/minute (decreasing)

rates of change (with respect time (t))

$$\frac{dw}{dt} = 2 \text{ ft/min}$$

$$\frac{dl}{dt} = -2 \text{ ft/min}$$

Step 2: Write equations (that show how the variables relate to each other)

Area = length x width

Perimeter = 2(length) + 2(width)

$$\text{Diagonal} = \sqrt{(\text{length})^2 + (\text{width})^2} \quad \text{(Pythagorean Theorem)}$$

Step 3: Solve using (implicit) differentiation

a) To find the change of area with respect to time,

$$\frac{dA}{dt} = \frac{dl}{dt} w + \frac{dw}{dt} l \quad \text{(product rule)}$$

$$\frac{dA}{dt} = -2 \text{ ft/min } (5) + 2 \text{ ft/min } (12) \quad \text{(substitution)}$$

$$\boxed{\frac{dA}{dt} = 14 \text{ feet/minute}}$$

b) To find the change in perimeter with respect to time,

$$\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$

$$\boxed{\frac{dP}{dt} = 2(-2 \text{ ft/min}) + 2(2 \text{ ft/min}) = 0 \text{ feet/minute}}$$

c) To find the change in each diagonal with respect to time,

$$\frac{dD}{dt} = \frac{1}{2}(l^2 + w^2)^{\frac{-1}{2}}\left(2l\frac{dl}{dt} + 2w\frac{dw}{dt}\right) \quad \text{(power rule/chain rule)}$$

$$\frac{dD}{dt} = \frac{1}{2}(144 + 25)^{\frac{-1}{2}}(-48 + 20)$$

$$\boxed{\frac{dD}{dt} = \frac{-28}{2(13)} = -\frac{14}{13} \text{ feet/minute}}$$

approx. -1.08

Step 4: Check for reasonableness

one minute ago: length = 14
width = 3
area = 42

$\triangle$ area = 18

now: length = 12
width = 5
area = 60

14 is in between!

$\triangle$ area = 10

one minute later: length = 10
width = 7
area = 70

That makes sense... As the lengths increase 2 feet each, the widths decrease 2 feet each. Although the shape is changing, the perimeter does not change.

one minute ago: length = 14
width = 3
diagonal $\approx$ 14.3

$\triangle$ diagonal = -1.3

now: length = 12
width = 5
diagonal = 13

-1.08 is in between!

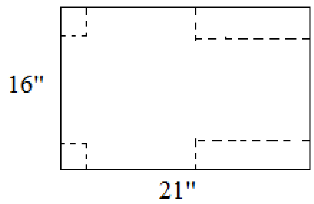
one minute later: length = 10
width = 7
diagonal $\approx$ 12.2

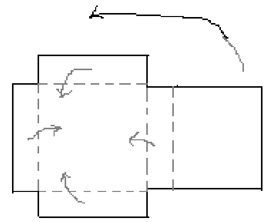
$\triangle$ diagonal = -.8

You are given a 16" x 21" cardboard sheet.
After cutting out the corners, you can fold up 3 of the sides.
Then, the fourth side will be folded up and extended over the other 3 to form a lid.

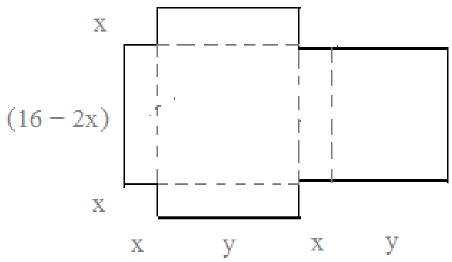
What are the dimensions of the enclosed box with the largest volume?

Step 1: Draw a diagram to visualize the question





Step 2: Label diagram, establish variables, and write equations



since $2x + 2y = 21$, $y = \frac{21 - 2x}{2}$

Volume = (length)(width)(height)

length = $(16 - 2x)$

height = x

width = $\frac{(21 - 2x)}{2}$

$$V = (16 - 2x)\left(\frac{(21 - 2x)}{2}\right)(x)$$

Step 3: Solve.

To find the maximum (or minimum) volume, find dV/dx and set it equal to 0...

$$V = (16x - 2x^2)\left(\frac{(21 - 2x)}{2}\right)$$

$$V = (8x - x^2)(21 - 2x)$$

$$V = 168x - 16x^2 - 21x^2 + 2x^3$$

$$V = 2x^3 - 37x^2 + 168x$$

$$\frac{dV}{dx} = 6x^2 - 74x + 168$$

then, set derivative equal to zero...

$$6x^2 - 74x + 168 = 0$$

$$3x^2 - 37x + 84 = 0$$

$$x = 3 \text{ or } 28/3$$

Step 4: Answer question and check solutions

If $x = 28/3$, height = 9.33
length = $(16 - 2(9.33)) = -2.66$

Impossible... $x \neq 28/3$

If $x = 3$, height = 3
length = $(16 + 2(3)) = 10$
width = $\frac{(21 + 2(3))}{2} = 7.5$

The dimensions of the box (with lid) are

10" x 7.5" x 3"

Check:

If x = 2, then dimensions are 12" x 8.5" x 2" 204 cubic inches
If x = 2.5, then dimensions are 11" x 8" x 2.5" 220 cubic inches
If x = 3, then dimensions are 10" x 7.5" x 3" 225 cubic inches ⇐
If x = 3.5, then dimensions are 9" x 7" x 3.5" 220.5 cubic inches
If x = 4, then dimensions are 8" x 6.5" x 4" 208 cubic inches

Examples: $\displaystyle\int \frac{x^2 - 2x - 2}{x^3 - 1}\,dx$

Factor the denominator, then decompose into partial fractions

Using Partial Fractions...

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx + C}{(x^2+x+1)}$$

Solve the rational equation with common denominators...

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A\,(x^2+x+1)}{(x-1)\,(x^2+x+1)} + \frac{(Bx + C)\,(x-1)}{(x^2+x+1)\,(x-1)}$$

$$x^2 - 2x - 2 = A\,(x^2+x+1) + (Bx + C)(x-1)$$

$$x^2 - 2x - 2 = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$$x^2 = (A + B)x^2$$

$$-2x = (A - B + C)x$$

$$-2 = (A - C)$$

Solve the system:

$A + B = 1$

$A - B + C = -2$

$A - C = -2$

$2A - B = -4$

$3A = -3$

$A = -1$

$B = 2$

$C = 1$

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx + C}{(x^2+x+1)}$$

$$= \frac{-1}{(x-1)} + \frac{2x + 1}{(x^2+x+1)}$$

$$\int \frac{x^2 - 2x - 2}{x^3 - 1}\,dx = \int \frac{-1}{(x-1)}\,dx + \int \frac{2x+1}{(x^2+x+1)}\,dx$$

$$-\ln|x - 1| + \ln|x^2 + x + 1| + C$$

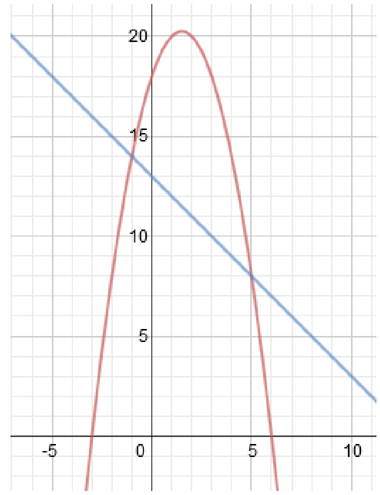
Integration: Volume of a Solid

Find the volume of the solid formed by the region bounded by

$$y = -x^2 + 3x + 18$$

$$x + y = 13$$

and revolved around the x-axis



When we graph the equations, we observe an upside down parabola that is intersected by a line.

The boundary (from left to right) are the points of intersection.

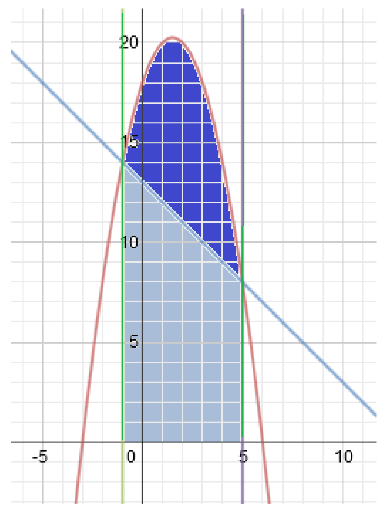
$$\begin{cases} y = -x^2 + 3x + 18 \\ y = -x + 13 \end{cases}$$ substitution

$$-x + 13 = -x^2 + 3x + 18$$ collect terms

$$x^2 - 4x - 5 = 0$$ factor

$$(x-5)(x+1) = 0$$ solve

$x = -1$ and $x = 5$



The outer radius will be from the parabola
The inner radius will be from the line

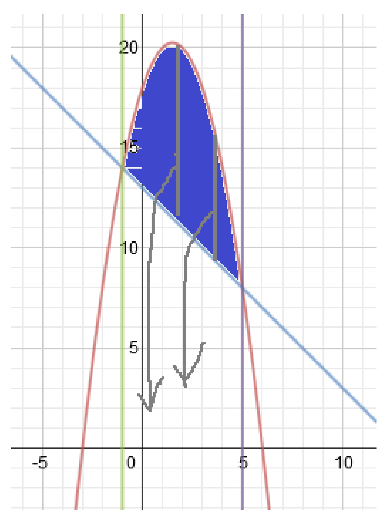
$$\int_{-1}^{5} -x^2 + 3x + 18 \, dx \quad - \quad \int_{-1}^{5} -x + 13 \, dx$$

area below the parabola — area below the line

$$\int_{-1}^{5} \pi(-x^2 + 3x + 18)^2 \, dx \quad - \quad \int_{-1}^{5} \pi(-x + 13)^2 \, dx$$

discs from the parabola — discs from the line

$$\pi \int_{-1}^{5} x^4 - 6x^3 - 27x^2 + 108x + 324 \, dx \quad - \quad \pi \int_{-1}^{5} x^2 - 26x + 169 \, dx$$

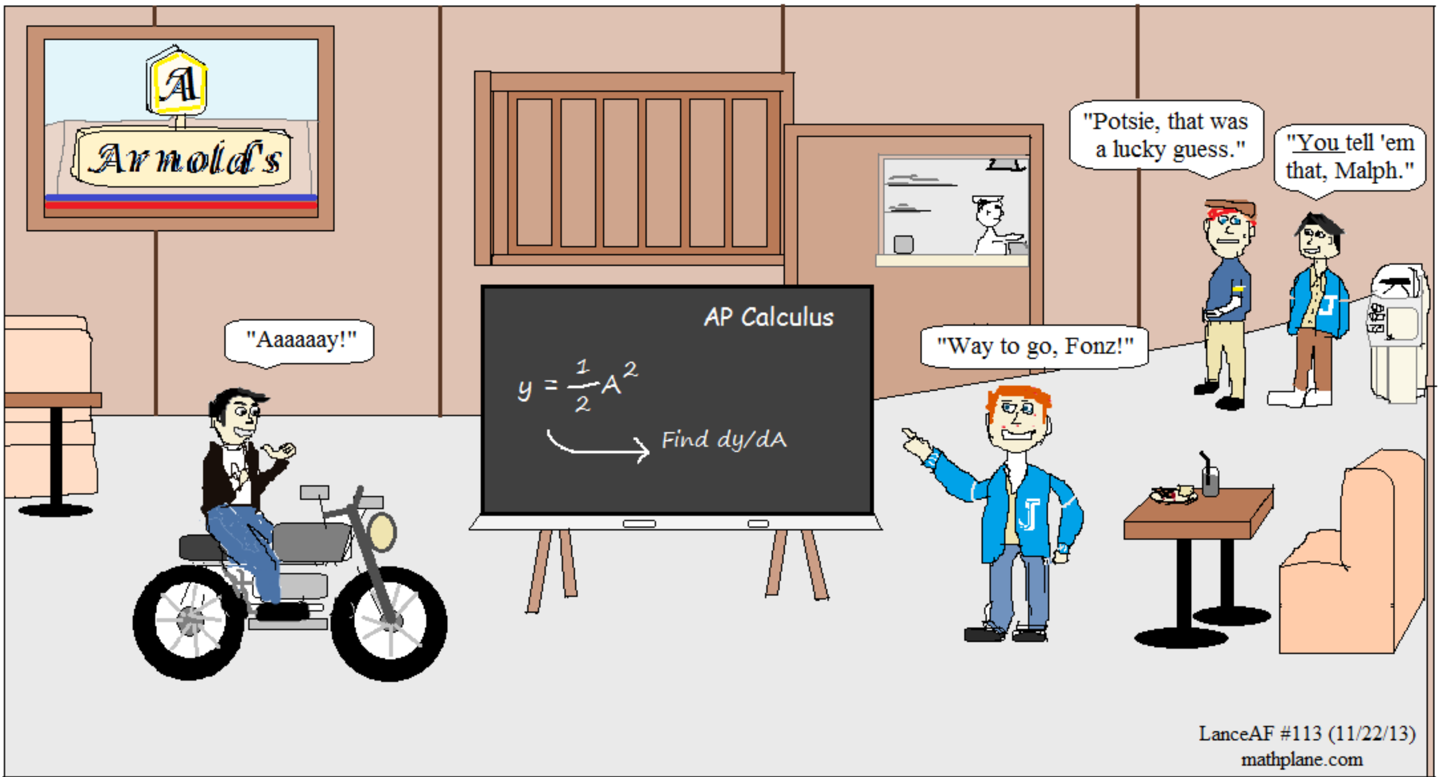


$$\pi \int_{-1}^{5} x^4 - 6x^3 - 28x^2 + 134x + 155 \, dx$$

$$\pi \left(\frac{x^5}{5} - \frac{3x^4}{2} - \frac{28x^3}{3} + 67x^2 + 155x \Big|_{-1}^{5} \right)$$

$$\pi \left(625 - 937.5 - 1166\tfrac{2}{3} + 1675 + 775 - \left(\frac{-1}{5} - \frac{3}{2} - \frac{-28}{3} + 67 - 155 \right) \right)$$

$$970.83 - (-80.36) = \boxed{1051.2\,\pi}$$



Despite Richie's help, Fonzie dropped out of Calculus.
(... although he did have some success with velocity and acceleration!)

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

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