VANDERBILT UNIVERSITY School of Engineering

Discrete Structures

CS 2212

(Fall 2020)

4 – Logic

Reminder and Recap …

Reminder: **ZyBook Assig. 1A** and **1B** due **Sep. 06** (11:59 PM)

Recap:

Predicate
(A statement with *variables*)

Quantifiers
(predicate to proposition)

Nested Quantifiers

Assign specific value to variable

Universal quantifier
(For all ∀)
Think of it as a conjunction

Existential quantifier
(There exist ∃)
Think of it as a disjunction

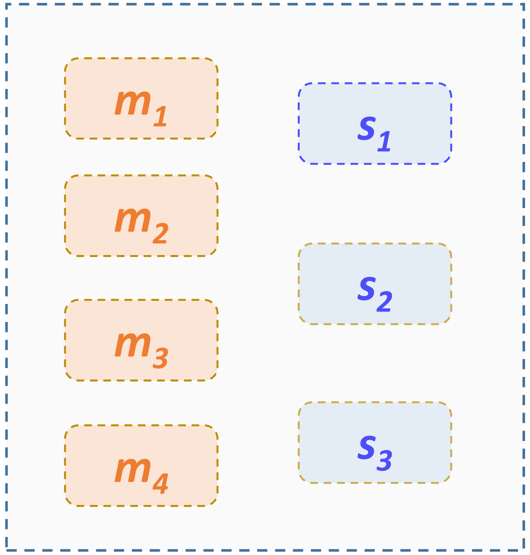
Today:

First, we will look at

English statements

Statements with nested quantifiers

Example - Nested Quantifiers



For every machine, there is a supervisor that operates it.

1. Identify variables
2. Predicate
3. Statement with nested quantifiers

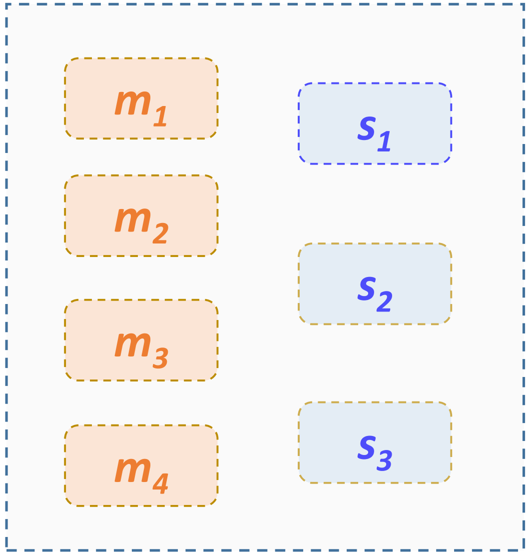
Machine: m

Supervisor: s

$P(m,s)$: m is operated by s.

$$\forall m \, \exists s \, P(m, s)$$

Example - Nested Quantifiers



Domain of m: $\{m_1, m_2, m_3, m_4\}$

Domain of s: $\{s_1, s_2, s_3\}$

$$\forall m \, \exists s \, P(m, s)$$

$$\equiv$$

$$(\exists s \, P(m_1, s)) \land (\exists s \, P(m_2, s)) \land (\exists s \, P(m_3, s)) \land (\exists s \, P(m_4, s))$$

Example - Nested Quantifiers

There is a student with A's in all courses.

Student: s

Course: c

$G(s,c)$: s scored A in c

There exists some s for which $G(s,c)$ is true for all c.

$$\exists s \, \forall c \, G(s,c)$$

Example - Nested Quantifiers

Domain of s: $\{s_1, s_2, s_3\}$

Domain of c: $\{c_1, c_2, c_3\}$

$$\exists s \, \forall c \, G(s,c)$$

$$\equiv$$

$$(\forall c \, G(s_1, c)) \lor (\forall c \, G(s_2, c)) \lor (\forall c \, G(s_3, c))$$

Nested Quantifiers Precedence

Operator	Precedence
$\forall$, $\exists$	1
$\neg$	2
$\wedge$ $\vee$	3 4
$\rightarrow$ $\leftrightarrow$	5 6

The quantifiers $\forall$ and $\exists$ have *higher* precedence than all the logical operators.

Nested Quantifiers Precedence

Predicate precedence with no presence of parentheses:

1. $\forall, \exists$
2. $\neg$
3. 4. $\wedge, \vee$
5. 6. $\rightarrow, \leftrightarrow$

Example:

$$\forall x \neg \exists y\, p(x, y) \rightarrow \forall x\, q(x)$$

$$\equiv (\forall x \neg \exists y\, p(x, y)) \rightarrow (\forall x\, q(x))$$

$$\equiv (\forall x \neg (\exists y\, p(x, y))) \rightarrow (\forall x\, q(x))$$

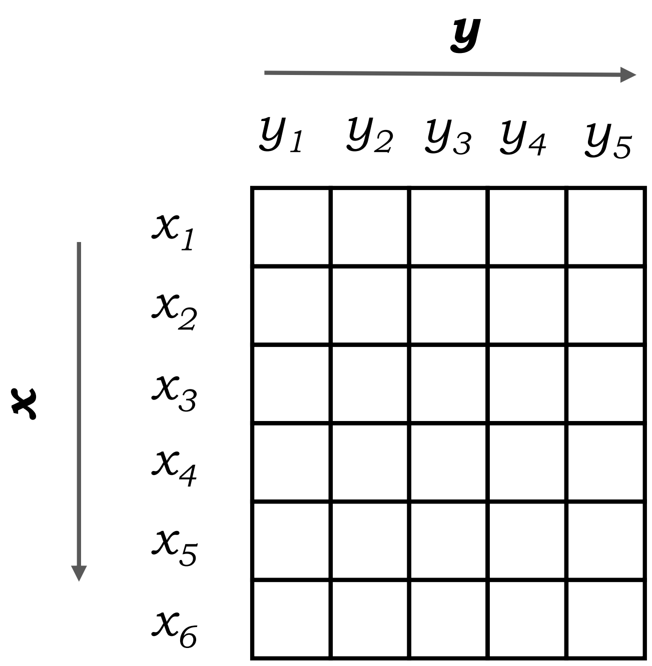
$$\equiv (\forall x (\neg (\exists y\, p(x, y)))) \rightarrow (\forall x\, q(x))$$

Nested Quantifiers

Two variable predicate: **P(*x*,*y*)**

Variable ***x*** = $\{x_1 , x_2 , \ldots , x_n\}$

Variable ***y*** = $\{y_1 , y_2 , \ldots , y_m\}$

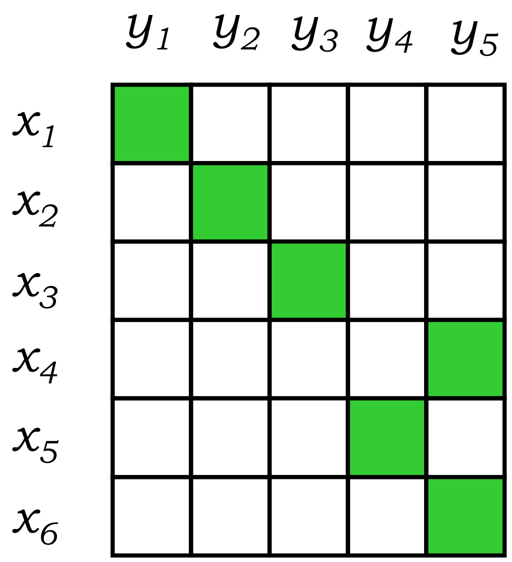


$P(x,y)$

$\forall x \, \exists y \, P(x, y)$

It means that in **every row**, there should be at least one true value (green block). We don't care where this true value is in the row, but each row must contain one.

Example:

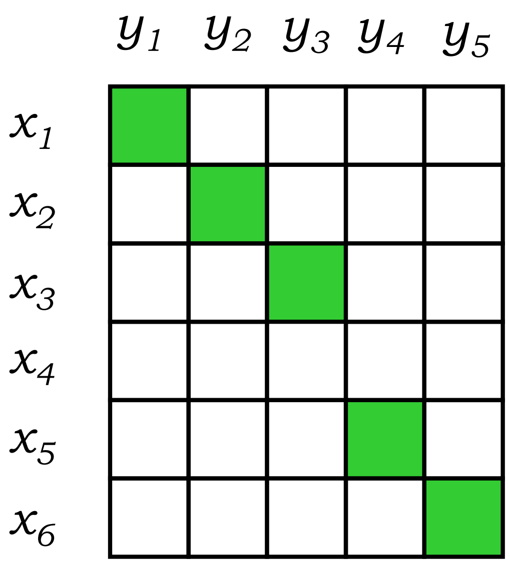


$\forall x \, \exists y \, P(x, y)$ is **true**

Nested Quantifiers

$\forall x \, \exists y \, P(x, y)$ is **false**

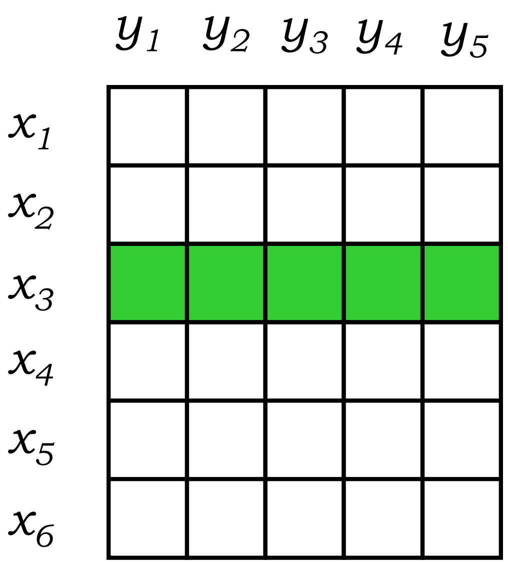
It means there is at least one **row** that **does not** have any true value.



$\neg \forall x \, \exists y \, P(x, y)$ is **true**

$\exists x \, \forall y \, P(x, y)$

We are looking for a **row** with all true values

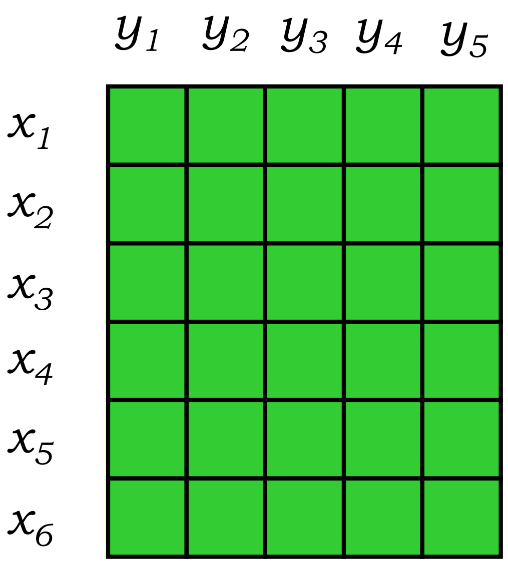


$\exists x \, \forall y \, P(x, y)$ is **true**

Nested Quantifiers

$\forall x \, \forall y \, P(x, y)$

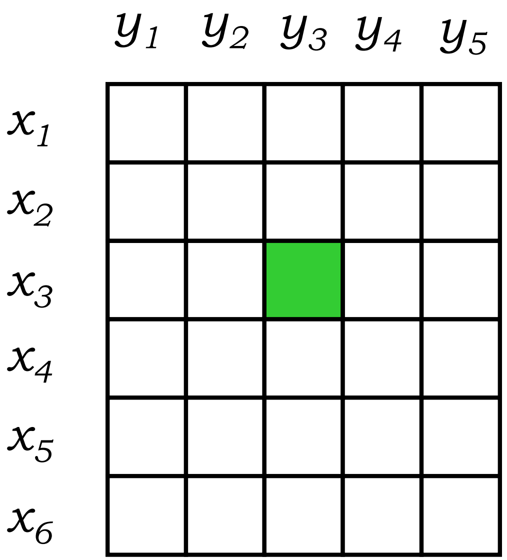
All blocks should be true



$\forall x \, \forall y \, P(x, y)$ is **true**

$\exists x \, \exists y \, P(x, y)$

We are looking for a **at least one block** to be true.



$\exists x \, \exists y \, P(x, y)$ is **true**

Nested Quantifiers

Statement	When True?	When False?
$\forall x \forall y\, P(x, y)$, $\forall y \forall x\, P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y\, P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y\, P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y\, P(x, y)$, $\exists y \exists x\, P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

(Summary)

Predicate Logic

Now, lets try to write these statements using quantifiers.

F(x,y): x can fool y.

(The domain consists of all people in the world).

Everybody can fool Fred.	$\forall x \, F(x, \text{Fred})$
Everybody can fool somebody.	$\forall x \, \exists y \, F(x, y)$
Nobody can fool everybody.	$\neg \exists x \, \forall y \, F(x, y)$
No one can fool himself/herself.	$\neg \exists x \, F(x, x)$

Predicate Logic

Lets see what does these statements mean?

$$\exists x\, \exists y\, ((x^2 = y^2) \wedge (x \neq y))$$

x is a real number
y is a real number

There exists two distinct real numbers whose squares are equal.

$$\exists x\, \exists y\, (P(x) \wedge P(y) \wedge (x \neq y))$$

There exist two distinct values for which statement P is true.

Predicate Logic

$$\exists x\, \exists y\, (P(x) \land P(y) \land (x \neq y)) \land \forall z\, (P(z) \rightarrow (z = x) \lor (z = y))$$

There exists two distinct values that make the statement true

Out of all the values, the statement is true for only x or y

Combining these two statements using 'and'

There are exactly two values that make the statement P true.

Predicate Logic

Every adult is married to exactly one adult. (ZyBook – 1.10.6)

$A(x)$: x is an adult

$M(x,y)$: x is married to y

$\forall x \, (A(x) \rightarrow \exists y \, M(x,y))$ → Every adult is married to someone

$\forall x \, \exists y \, (A(x) \rightarrow M(x,y))$

$\forall x \, \exists y \, (A(x) \rightarrow (M(x,y) \wedge (\forall z \, M(x,z) \rightarrow (z=y))))$

$\forall z \, M(x,z) \rightarrow (z=y)$ → If an adult is married, he/she is only married to one person

$\forall x \, \exists y \, \forall z \, (A(x) \rightarrow (M(x,y) \wedge (M(x,z) \rightarrow (z=y))))$

Note: Do you note any difference with the solution in the book? How can you justify that both solutions are correct?

Predicate Logic

So far, we have seen,

How to formally and systematically write statements (which provide some information) using propositions and predicates.

Next, we will see,

How to **reason** and formally **prove** (or **disprove**) some statement/result from a given set of statements.

Logic and Predicates: Proofs

Example:

I eat spinach (S) or ice cream (I). If I study logic (L) then I will pass the exam (E). If I eat ice cream, then I will study logic. If I eat spinach, then I will play golf (G). I failed the exam.

Therefore, I played golf.

Logic and Predicates: Proofs

Premise / Hypothesis

1. I eat spinach (S) or ice cream (I). — $(S \lor I)$
2. If I study logic (L) then I will pass the exam (E) — $(L \to P)$
3. If I eat ice cream, then I will study logic. — $(I \to L)$
4. If I eat spinach, then I will play golf (G). — $(S \to G)$
5. I failed the exam. — $\neg P$

Conclusion

Therefore, I played golf. — G

$$((S \lor I) \land (L \to P) \land (I \to L) \land (S \to G) \land \neg P) \to G$$

Argument

Logic and Predicates: Proofs

$$((S \vee I) \wedge (L \rightarrow P) \wedge (I \rightarrow L) \wedge (S \rightarrow G) \wedge \neg P) \rightarrow G$$

- So, the goal is to **simplify** statements (arguments) like these and show if the conclusion holds or not.
- How can we simplify?
- By using **rules of inference.**
- Lets see some of them.

Logic and Predicates: Rules of Inference

$$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$$

Conjunction

$$\begin{array}{l} p \\ \hline \therefore p \lor q \end{array}$$

Addition

$$\begin{array}{l} p \land q \\ \hline \therefore p \end{array}$$

Simplification

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

Disjunctive syllogism

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Modus ponens

$$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$

Modus tollens

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Hypothetical syllogism

$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$$

Resolution

Logic and Predicates: Proofs

Problem: I eat spinach (S) or ice cream (I). If I study logic (L) then I will pass the exam (E). If I eat ice cream, then I will study logic. If I eat spinach, then I will play golf (G). I failed the exam. Therefore, I played golf.

Argument:

$$((S \vee I) \wedge (L \rightarrow E) \wedge (I \rightarrow L) \wedge (S \rightarrow G) \wedge \neg E) \rightarrow G$$

Logic and Predicates: Proofs

Argument: $((S \vee I) \wedge (L \rightarrow E) \wedge (I \rightarrow L) \wedge (S \rightarrow G) \wedge \neg E) \rightarrow G$

	Statements	Why?
1	$S \vee I$	Premise
2	$L \rightarrow E$	Premise
3	$I \rightarrow L$	Premise
4	$S \rightarrow G$	Premise
5	$\neg E$	Premise
6	$\neg L$	2, 5, Modus tollens
7	$\neg I$	3, 6, Modus tollens
8	S	1, 7 , Disjunctive syllogism
9	G	4, 8, Modus Ponens
10	QED	1 - 9

Logic and Predicates: Proofs

Example: Prove that the argument with premises $\mathbf{A \vee C \rightarrow D}$, $\mathbf{\neg B}$, $\mathbf{A \vee B}$ and with the conclusion $\mathbf{D}$ is valid.

What we're really being asked to do is prove…

$(A \vee C \rightarrow D) \wedge \neg B \wedge (A \vee B) \rightarrow D$ **is true.**

Line	Statements	Why?
1	$A \vee C \rightarrow D$	Premise
2	$\neg B$	Premise
3	$A \vee B$	Premise
4	A	2, 3, Disjunctive Syll.
5	$A \vee C$	4, Addition
6	D	1, 5, Modus Ponens
7	QED	1-6

$(A \vee C \rightarrow D) \wedge \neg B \wedge (A \vee B)$ — A is true

$A \vee C \rightarrow D$ — $(A \vee C)$ is true

D

Proofs

Just a reminder.

$\equiv$	vs.	$\rightarrow$

Equivalence

$\mathbf{P \equiv Q}$

Conditional

$\mathbf{P \rightarrow Q}$

So far, we have seen proofs in two contexts:

1. Proving that two statements are equivalent (**equivalence proofs**).
2. Proving that if a statement is true, then it implies some conclusion (**conditional proofs**).

Indirect Proofs*

Our goal is to prove: **$A \rightarrow B$**

- So far, we have seen how to **use inference rules** and show that hypotheses on L.H.S imply the conclusion on the R.H.S.
- There is an another interesting way – **Indirect proofs**.
- First recall two facts:

1. A proposition cannot be true and false at the same time.

   $(A \land \neg A) = \text{False}$ (a contradiction).

2. If $(A \rightarrow B)$ then $(\neg B \rightarrow \neg A)$. Recall modus tollens.

   In words, if A is true, we know B is true. B is necessary for A. Consequently, if B is false, A must be false. Hence, $(\neg B \rightarrow \neg A)$.

(* Not in ZyBook)

Indirect Proofs - Approach

Our goal is to prove: **$A \rightarrow B$**

- May be it is difficult to “simplify” A and show A implies B.
- So, we use an **alternate approach (indirect proof)**.

We assume B is not true, that is $\neg$ B.

Then we prove using rules of inference that $\neg B \rightarrow \neg A$

(May be showing $\neg B \rightarrow \neg A$ is easier and straightforward as compared to showing $A \rightarrow B$.)

But we know (for sure) that A is true as it is a given premise. However, in the above step we showed that A is false if I assume that B false.

Since A can’t be true and false at the same time, my assumption that B is false is wrong.

Thus, B is true if A is true.

Hence $A \rightarrow B$

Indirect Proofs

Summary:

Prove: $A \rightarrow B$

1. Assume: $\neg B$
2. Show: $\neg B \rightarrow \neg A$
3. Observe: A is a premise, and $(A \wedge \neg A) = \text{False}$
4. Therefore: $\neg B$ is false
5. Hence: B is true, and $A \rightarrow B$

Indirect Proofs - Example

Prove: If $3n+2$ is odd, then n is odd

P: $3n+2$ is odd

Q: n is odd

Show: $P \rightarrow Q$

1.	P	Premise
2.	$\neg$ Q (n is even).	Assumption
3.	$n = 2k$	By the definition of even numbers
4.	$3n+2 = 3(2k) + 2$	Replacing n in $(3n+2)$
5.	$2(3k+1)$	Simplifying line 3
6.	$2(3k+1)$ is even	By the definition of even numbers
7.	$\neg$ P	From line 5
8.	$P \wedge \neg P$ = False	1,7, Contradiction
9.	$P \rightarrow Q$	QED.

Indirect Proofs - Example

Lets look at another example of indirect proofs.

Prove: $(\mathbf{A} \lor \mathbf{C} \rightarrow \mathbf{D}) \land \neg \mathbf{B} \land (\mathbf{A} \lor \mathbf{B}) \rightarrow \mathbf{D}$

Previously, we proved it using a direct approach. Now, we use an indirect approach.

Prove: $(A \vee C \rightarrow D) \wedge \neg B \wedge (A \vee B) \rightarrow D$

Line	Statements	Why?
1	$A \vee C \rightarrow D$	Premise
2	$\neg B$	Premise
3	$A \vee B$	Premise
4	$\neg D$	Assumption
5	$\neg (A \vee C)$	1, 4, Modus tollens
6	$\neg A \wedge \neg C$	5, DeMorgan's Law
7	$\neg A$	6, Conjunction
8	$\neg A \wedge \neg B$	2,7
9	$\neg (A \vee B)$	8, DeMorgans Law
10	$\neg(A \vee C \rightarrow D) \vee \neg(\neg B) \vee \neg (A \vee B)$	9, Disjunction
11	$\neg((A \vee C \rightarrow D) \wedge (\neg B) \wedge (A \vee B))$	10, DeMorgans Law
12	$(A \vee C \rightarrow D) \wedge \neg B \wedge (A \vee B)$	1,2,3 (Hypotheses)
13	False	11,12, Contradiction
14	D	13