Sample Test 2 Solutions

1. Do the following converge (explain)?

(1.1) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^4+1}$,

Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$. Since $\ln n < n$ for $n \geq 1$, then $\frac{\ln n}{n^4+1} < \frac{n}{n^4+1}$. This implies that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^4+1} < \sum_{n=1}^{\infty} \frac{n}{n^4+1}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges (direct comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ $p = 3$) then by the direct comparison test (DCT), the original series converges.

(1.2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+1}$,

Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p series with $p = 3$) then by the limit comparison test (LCT), the original series converges.

(1.3) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{n}\right)^n$,

Taking the limit $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{n}\right) = \frac{1}{2} < 1$ then by the n^{th} root test, the original series converges.

(1.4) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$,

Consider $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!} / \frac{e^n}{(n)!} = \lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!} \cdot \frac{(n)!}{e^n}$
$\displaystyle= \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$ so by ratio test, the series converges

(1.5) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$,

Since $\ln(n+1) < n+1$ for $n \geq 1$ then $\displaystyle\frac{1}{n+1} < \frac{1}{\ln(n+1)}$ for $n \geq 1$ and since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)}$ (harmonic) diverges, then by the DCT, original series does as well.

(1.6) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$,

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ then $\displaystyle\lim_{n\to\infty} \frac{1}{n(n+1)} / \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^2}{n(n+1)} = 1$, and since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2$) then by the limit comparison test (LCT) the original series converges.

(1.7) $\sum_{n=1}^{\infty} \frac{n-1}{n+1}$,

Since $\lim_{n\to\infty} \frac{n-1}{n+1} = 1$, then by the n^{th} term test for divergence, the series diverges.

(1.8) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$,

Consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)!^2} / \frac{(2n)!}{(n)!^2} = \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \cdot \frac{(n+1)!^2}{n!^2}$
$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4 > 1$ so by ratio test, the series diverges

(1.9) $\sum_{n=2}^{\infty} \frac{1}{\ln^2(n)}$,

Since $\ln n < n$ for $n \geq 1$ then $\ln^2 n < n \ln n$ for $n \geq 1$ which gives $\frac{1}{n \ln n} < \frac{1}{\ln^2 n}$ for $n \geq 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges, (see next question) then by the direct comparison test, original series does as well.

(1.10) $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$,

Let $f(x) = \frac{1}{x \ln x}$. Clearly $f(x) > 0$ and $f'(x) = -\frac{\ln x + 1}{(x \ln x)^2}$ for $x \geq 3$ showing that $f(x)$ is decreasing so that the integral test may be used. Consider

$$\int_3^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty} \int_3^b \frac{dx}{x \ln x} = \lim_{b\to\infty} \ln \ln x|_3^b = \infty.$$

Since the integral diverges, then by the integral test, the series does as well.

(1.11) $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$.

Compare with $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Then $\lim_{n\to\infty} \frac{1}{2^n} / \frac{1}{2^n+1} = \lim_{n\to\infty} \frac{2^n+1}{2^n} = 1$ and since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (geometric series with $r = 1/2$), the original series converges by the LCT.

2. Determine whether the following series converge absolutely, conditionally or diverge

(2.1) $\sum_{n=1}^{\infty} \frac{(-1)^n (n-1)}{n+1}$, Since $\lim_{n\to\infty} \frac{(-1)^n(n-1)}{n+1} = (-1)^n \neq 0$ this series diverges.

(2.2) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n(n+1)}}$,

We first consider $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$ and by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$ shows that we do not have absolute convergence. So we check the two conditions for conditional convergence. If we let $a_n = \frac{1}{\sqrt{n(n+1)}}$, then clearly

$$\lim_{n\to\infty} \frac{1}{\sqrt{n(n+1)}} = 0.$$

Next, we need to show $a_{n+1} < a_n$. We could show

$$\frac{1}{\sqrt{(n+1)(n+2)}} \quad \overset{?}{\leq} \quad \frac{1}{\sqrt{n(n+1)}},$$

but is easier to show that if

$$f(x) = \frac{1}{\sqrt{x(x+1)}} \quad \text{then} \quad f'(x) = -\frac{2x+1}{2(x^2+x)^{3/2}} < 0 \quad \text{for } x \geq 1$$

so by the alternating series test (AST), the series converges conditionally.

(2.3) $\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$,

For this question we will use the ratio test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} / \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} =$$
$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \mathrm{e} > 1$$

so by ratio test, the series diverges

(2.4) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n+3^n}$,

We first consider $\sum_{n=1}^{\infty} \frac{1}{2^n+3^n}$ and compare with $\sum_{n=1}^{\infty} \frac{1}{3^n}$. By the LCT

$$\lim_{n\to\infty} \frac{1}{3^n} / \frac{1}{2^n+3^n} = \lim_{n\to\infty} \frac{2^n+3^n}{3^n} = \lim_{n\to\infty} 1 + \left(\frac{2}{3}\right)^n = 1,$$

and since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (geometric series $r = 1/3$), the original series converges absolutely.

(2.5) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$,

We first consider $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ and compare with $\sum_{n=1}^{\infty} \frac{1}{n}$. By the LCT

$$\lim_{n\to\infty} \frac{1}{n} / \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n^2+1}{n^2} = 1,$$

which show that original series doesn't converge absolutely since we compared with the harmonic series that diverges. If we let $a_n = \frac{n}{n^2+1}$, then clearly

$$\lim_{n\to\infty} \frac{n}{n^2+1} = 0.$$

Next, we need to show $a_{n+1} < a_n$. If we let

$$f(x) = \frac{x}{x^2+1} \quad \text{then} \quad f'(x) = \frac{-x^2+1}{(x^2+1)^2} < 0 \ \text{ for } \ x > 1$$

so by the alternating series test (AST), the series converges conditionally.

(2.6) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$.

Since $\lim_{n\to\infty} \frac{(-1)^n n}{n+1} = (-1)^n \neq 0$ this series diverges.

3. Determine the interval of convergence of the following.

3(i) $\sum_{n=1}^{\infty} \frac{2^n x^n}{\sqrt{n+1}}$,

Choosing

$$a_n = \frac{2^n x^n}{\sqrt{n+1}}$$

then

$$\lim_{x\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{x\to\infty} \left| \frac{2^{n+1} x^{n+1}}{\sqrt{n+2}} / \frac{2^n x^n}{\sqrt{n+1}} \right| = \lim_{x\to\infty} 2 \frac{\sqrt{n+1}}{\sqrt{n+2}} |x| = 2|x| < 1$$

So $|x| < \frac{1}{2}$ or $-\frac{1}{2} < x < \frac{1}{2}$. Checking the endpoints gives

$$x = -\frac{1}{2} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}, \quad \text{which converges by AST}$$

$$x = \frac{1}{2} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}, \quad \text{which diverges by DCT with } p \text{ series } (p = 1/2)$$

Therefore the interval of convergence is $-\frac{1}{2} \leq x < \frac{1}{2}$.

$4(ii) \quad \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2},$

Choosing

$$a_n = \frac{(-1)^n x^{2n}}{2^{2n} n!^2}$$

then

$$\lim_{x\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{x\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{2^{2n+2}(n+1)!^2} \Big/ \frac{(-1)^n x^{2n}}{2^{2n} n!^2} \right| = \lim_{x\to\infty} \frac{1}{4(n+1)^2} \left| x^2 \right| = 0 < 1$$

so series converges for all x.

$4(iii) \quad \sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^2},$

Choosing

$$a_n = \frac{(2x-1)^n}{n^2}$$

then

$$\lim_{x\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{x\to\infty} \left| \frac{(2x-1)^{n+1}}{(n+1)^2} \Big/ \frac{(2x-1)^n}{n^2} \right| = \lim_{x\to\infty} \frac{n^2}{(n+1)^2} |2x-1| = |2x-1| < 1$$

So $|2x-1| < 1$ or $-1 < 2x-1 < 1$ or $0 < 2x < 2$ or $0 < x < 1$. Checking the endpoints gives

$$x = 0 \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \quad \text{which converges absolutely, it's a } p \text{ series}$$

$$x = 1 \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{which converges, it's a } p \text{ series}$$

Therefore the interval of convergence is $0 \leq x \leq 1$.

4. Calculate the n^{th} degree Taylor polynomial with remainder for the following. Expand about the point $x = c$

(4.1) $f(x) = \mathrm{e}^x, \ \ c = 0, \ \ \mathrm{n} = 2$

$$\begin{array}{rcll} f(x) & = & \mathrm{e}^x & f(0) = 1, \\ f'(x) & = & \mathrm{e}^x & f'(0) = 1, \\ f''(x) & = & \mathrm{e}^x & f''(0) = 1, \\ f'''(x) & = & \mathrm{e}^x & \text{for the remainder}, \end{array}$$

The Taylor polynomial is

$$\begin{aligned} P_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2, \\ &= 1 + x + \frac{x^2}{2!}, \end{aligned}$$

The remainder is given by

$$R_2(x) = \frac{\mathrm{e}^z}{3!}x^3,$$

for $0 < z < x$ or $x < z < 0$

(4.2) $f(x) = \sin x, \ \ c = \frac{\pi}{2} \ \ n = 4,$

In this example, we need only construct P_4 .

$$\begin{array}{ll} f(x) = \sin x & f(\frac{\pi}{2}) = 1, \\ f'(x) = \cos x & f'(\frac{\pi}{2}) = 0, \\ f''(x) = -\sin x & f''(\frac{\pi}{2}) = -1, \\ f'''(x) = -\cos x & f'''(\frac{\pi}{2}) = 0, \\ f^{(4)}(x) = \sin x & f^{(4)}(\frac{\pi}{2}) = 1, \end{array}$$

The Taylor polynomial is

$$\begin{aligned} P_4(x) &= f(\frac{\pi}{2}) + \frac{f'(\frac{\pi}{2})}{1!}(x - \frac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2!}(x - \frac{\pi}{2})^2 + \cdots + \frac{f^{(4)}(\frac{\pi}{2})}{4!}(x - \frac{\pi}{2})^4, \\ &= 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4, \end{aligned}$$

The remainder is given by

$$\begin{aligned} R_4(x) &= \frac{f^{(5)}(z)}{5!}\left(x - \frac{\pi}{2}\right)^5, \\ &= \frac{\cos z}{5!}\left(x - \frac{\pi}{2}\right)^5, \end{aligned}$$

for $\frac{\pi}{2} < z < x$ or $x < z < \frac{\pi}{2}$.

(4.3) $f(x) = \ln(x+1), \quad c = 0, \quad \text{n = 3}$

$$\begin{aligned}
f(x) &= \ln(x+1) & f(0) &= 0, \\
f'(x) &= \frac{1}{x+1} & f'(0) &= 1, \\
f''(x) &= \frac{-1}{(x+1)^2} & f''(0) &= -1, \\
f'''(x) &= \frac{2}{(x+1)^3} & f''(0) &= 2, \\
f^{(4)}(x) &= \frac{-3!}{(x+1)^4} \text{ for the remainder.}
\end{aligned}$$

The Taylor polynomial is

$$\begin{aligned}
P_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{n!}x^3, \\
&= 0 + x - \frac{1}{2!}x^2 + \frac{2!}{3!}x^3, \\
&= x - \frac{x^2}{2} + \frac{x^3}{3},
\end{aligned}$$

The remainder is given by

$$\begin{aligned}
R_3(x) &= \frac{f^{(4)}(z)}{4!}x^4, \\
&= -\frac{3!}{(z+1)^4}\frac{x^4}{4!},
\end{aligned}$$

for $0 < z < x$ or $x < z < 0$.

(4.4) $f(x) = \frac{1}{2-x}, \quad c = 0, \quad n = 3.$

In this case we only need P_3. The derivatives are:

$$\begin{aligned}
f(x) &= \frac{1}{(2-x)}, & f(0) &= \frac{1}{2}, \\
f'(x) &= \frac{1}{(2-x)^2}, & f'(0) &= \frac{1}{2^2}, \\
f''(x) &= \frac{2}{(2-x)^3}, & f''(0) &= \frac{2}{2^3}, \\
f'''(x) &= \frac{3!}{(2-x)^4}, & f'''(0) &= \frac{3!}{2^4}, \\
f^{(4)}(x) &= \frac{4!}{(2-x)^4}, & & \text{remainder},
\end{aligned}$$

The Taylor polynomial is

$$\begin{aligned}
P_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3, \\
&= \frac{1}{2} + \frac{x}{2^2} + \frac{2!}{2^3}\frac{x^2}{2!} + \frac{3!}{2^4}\frac{x^3}{3!}, \\
&= \frac{1}{2} + \frac{x}{2^2} + \frac{x^2}{2^3} + \frac{x^3}{2^4},
\end{aligned}$$

The remainder is given by

$$\begin{aligned}
R_3(x) &= \frac{f^{(4)}(z)}{4!}x^4, \\
&= \frac{1}{(2-z)^5}x^4.
\end{aligned}$$

for $0 < z < x$ or $x < z < 0$.