[5] L. Cohen, *Time-Frequency Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1995.
[6] L. Cohen and C. Lee, "Instantaneous bandwidth," in *Time-Frequency Signal Analysis*, B. Boashash, Ed. London, U.K.: Longman Cheshire, 1992.
[7] L. Cohen, "Distributions concentrated along the instantaneous frequency," *SPIE, Adv. Signal Proc. Alg., Arch., Implement.*, vol. 1348, pp. 149–157, 1990.
[8] A. Goldenshluger and A. Nemirovski, "On spatial adaptive estimation of nonparametric regression," Dept. Elec. Eng., Technion, Haifa, Israel, Res. Rep., 5/94, Nov. 1995.
[9] ——, "Adaptive de-noising of signals satisfying differential inequalities," *IEEE Trans. Inform. Theory*, vol. 43, pp. 873–889, May 1997.
[10] V. Katkovnik, "Nonparametric estimation of instantaneous frequency," *IEEE Trans. Inform. Theory*, vol. 43, pp. 183–189, Jan. 1997.
[11] ——, "Local polynomial approximation of the instantaneous frequency: Asymptotic accuracy," *Signal Processing*, vol. 52, no. 3, pp. 343–356, 1996.
[12] ——, "Adaptive local polynomial periodogram for time-varying frequency estimation," in *Proc. IEEE-SP IS-TFTSA*, Paris, France, June 1996, pp. 329–332.
[13] V. Katkovnik and LJ. Stanković, "Instantaneous frequency estimation using the Wigner distribution with varying and data-driven window length," *IEEE Trans. Signal Processing*, vol. 46, pp. 2315–2325, Sept. 1998.
[14] D. Petranović, S. Stanković, and LJ. Stanković, "Special purpose hardware for time-frequency analysis," *Electron. Lett.*, vol. 33, no. 6, pp. 464–466, Mar. 1997.
[15] K. J. Ray Liu, "Novel parallel architectures for short-time Fourier transform," *IEEE Trans. Circuits Syst.*, vol. 40, pp. 786–789, Dec. 1993.
[16] P. Rao and F. J. Taylor, "Estimation of the instantaneous frequency using the discrete Wigner distribution," *Electron. Lett.*, vol. 26, pp. 246–248, 1990.
[17] B. Ristic and B. Boashash, "Relationship between the polynomial and the higher order Wigner–Ville distribution," *IEEE Signal Processing Lett.*, vol. 2, pp. 227–229, Dec. 1995.
[18] LJ. Stanković and S. Stanković, "On the Wigner distribution of discrete-time noisy signals with application to the study of quantization effects," *IEEE Trans. Signal Processing*, vol. 42, pp. 1863–1867, July 1994.
[19] ——, "An analysis of the instantaneous frequency presentation using time-frequency distributions—Generalized Wigner distribution," *IEEE Trans. Signal Processing*, vol. 43, pp. 549–552, Feb. 1995.
[20] LJ. Stanković, "A method for improved distribution concentration in the time-frequency analysis of multicomponent signals the $L$-Wigner distribution," *IEEE Trans. Signal Processing*, vol. 43, pp. 1262–1268, May 1995.
[21] ——, "Highly concentrated time-frequency distribution: Pseudo quantum signal representation," *IEEE Trans. Signal Processing*, vol. 45, pp. 543–552, Mar. 1997.
[22] ——, "A multitime definition of the Wigner higher order distribution: L–Wigner distribution," *IEEE Signal Processing Lett.*, vol. 1, pp. 106–109, July 1994.
[23] ——, "L-class of time-frequency distributions," *IEEE Signal Processing Lett.*, vol. 3, pp. 22–25, Jan. 1996.

# Polynomial Cancellation Coding and Finite Differences

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***Abstract*—We give here a mathematical context for polynomial cancellation coding, proposed recently to reduce intercarrier interference in orthogonal frequency division multiplexing (OFDM). In particular, we analyze polynomial cancellation coding (PCC) in terms of finite differences.**

***Index Terms*—Finite differences, orthogonal frequency division multiplexing, polynomial cancellation coding.**

## I. Introduction

A technique has been proposed recently [1] which provides several benefits for orthogonal frequency division multiplexing (OFDM). In particular, this technique, termed *polynomial cancellation coding* (PCC), reduces the intercarrier interference (ICI) due to frequency shift between transmitter and receiver [2]. Further, PCC has been discussed in the context of reduction to out-of-band power and intersymbol interference in OFDM systems [3]. Used in its simplest form, PCC achieves these advantages at the cost of bandwidth efficiency. However, the advantages can be retained while maintaining, or even increasing, bandwidth efficiency if the symbol periods of the PCC coded data are overlapped [3] and an equalizer is used at the receiver to recover the transmitted data from the overlapped symbols.

The main idea of PCC is to map each complex number which is to be transmitted onto a group of $k$ subcarriers, with appropriate weightings, rather than to a single subcarrier. In a previous article [2], the weightings have been given as the coefficients of the polynomial $(1-x)^{k-1}$. It has been claimed that if the same weightings are applied in decoding the received signal, polynomial variation of order $(2k-3)$ in the ICI is canceled.

In this correspondence, we explain how ICI cancellation is achieved by relating PCC to finite-difference techniques, well known in numerical analysis. In Section II, we give mathematical expressions for polynomial cancellation coding as described in [2]. In Section III we list some standard results in finite difference theory, and recast our equations in this language. Finally, in Section IV we discuss the reduced ICI obtained by use of PCC.

## II. Polynomial Cancellation Coding

In the $i$th symbol period (length $T$) of an OFDM communications system, the complex numbers $a_{0,i}, \cdots, a_{N-1,i}$ modulate the $N$ subcarriers. If we assume an ideal channel, and the local oscillator at the

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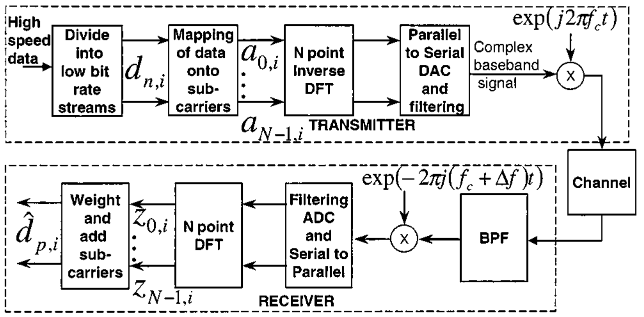



Fig. 1. Block diagram of an OFDM system with PCC with $N$ subcarriers. In each symbol period $i$, $\lfloor N/k \rfloor$ complex values are transmitted ($d_{n,i}$) and received ($\hat{d}_{p,i}$).

receiver is set at frequency $\Delta f$ above the correct frequency $f_c$, the $m$th output signal is

$$z_{m,i} = \sum_{l=0}^{N-1} c_{l-m} a_{l,i} \tag{1}$$

where the coefficients $c_{l-m}$ resulting from the frequency shift are [2]

$$\begin{aligned} c_{l-m} &= \frac{1}{N} \sum_{\ell=0}^{N-1} \exp\left(\frac{j2\pi\ell(l-m+\Delta f T)}{N}\right) \\ &= \frac{1}{N} \frac{\sin(\pi(l-m+\Delta f T))}{\sin(\pi(l-m+\Delta f T)/N)} \\ &\quad \times \exp\left(\frac{j\pi(N-1)(l-m+\Delta f T)}{N}\right) \end{aligned} \tag{2}$$

in which we suppress the simple rotation due to phase offset. The ICI affecting the $m$th complex value is made up of the terms in the sum in (1) for which $l \neq m$.

In PCC [1], each complex value $d_{n,i}$ to be transmitted is copied, with weighting, onto $k$ subcarriers. (Thus for a system with $N$ subcarriers, at most $\lfloor N/k \rfloor$ values can be transmitted in this way per symbol $i$, where $\lfloor x \rfloor$ denotes the integer part of $x$.) A diagram of an OFDM system with PCC is given in Fig. 1

The subcarriers $0, 1, \cdots, k-1$ are used to transmit the first complex value, and in general the subcarriers $\{nk+r\}_{r=0}^{k-1}$ carry the $(n+1)$th value, where $n = 0, 1, \cdots, \lfloor N/k \rfloor - 1$. The weightings [2] within the subcarrier groups are taken from the coefficients in the binomial expansion of

$$(1-x)^{k-1} = \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} x^r \tag{3}$$

so that the weighted copies of the complex number $d_{n,i}$ are

$$a_{nk+r,i} = (-1)^r \binom{k-1}{r} d_{n,i}, \qquad r = 0, 1, \cdots, k-1. \tag{4}$$

Thus the $m$th output signal in symbol $i$ is (from (1) and (4))

$$\begin{aligned} z_{m,i} &= \sum_{n=0}^{\lfloor N/k \rfloor -1} \sum_{r=0}^{k-1} c_{nk+r-m} a_{nk+r,i} \\ &= \sum_{n=0}^{\lfloor N/k \rfloor -1} d_{n,i} \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} c_{nk+r-m}. \end{aligned} \tag{5}$$

The outputs are combined in groups of size $k$ to decode the $p$th complex number $\hat{d}_{p,i}$, where $p = 0, 1, \cdots, \lfloor N/k \rfloor - 1$, from the weighted sum of $\{z_{pk+s,i}\}_{s=0}^{k-1}$

$$\hat{d}_{p,i} = \sum_{s=0}^{k-1} (-1)^s \binom{k-1}{s} z_{pk+s,i}. \tag{6}$$

If we now write $s' = k-1-s$, substitute from (5), and make the further change of variables $r' = r + s'$ so that the identity (see, e.g., [4]) for binomial coefficients

$$\sum_{m=0}^{n} \binom{M}{m} \binom{N}{n-m} = \binom{M+N}{n}$$

can be used, we obtain

$$\hat{d}_{p,i} = \sum_{n=0}^{\lfloor N/k \rfloor -1} d_{n,i} \left\{ \sum_{r'=0}^{2k-2} (-1)^{r'+k-1} \binom{2k-2}{r'} c_{k(n-p+1)-r'-1} \right\} \tag{7}$$

as the $p$th decoded complex value. Combinations of terms with the structure in the brackets in (7) are well known in the field of numerical analysis, and we next rewrite our formulas in terms of finite differences.

## III. Finite-Difference Formulas

In numerical analysis, finite-difference techniques are used to interpolate or extrapolate from tabulated data, and in numerical integration and differentiation. We need only consider techniques appropriate to evenly spaced data. We give here the results we need, and refer the reader to [5] or a similar work for further details.

Given a function $f(x)$, the (first) *central difference* is

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \tag{8}$$

where the step size $h$ is suppressed if it is understood to be 1. Successive central differences are defined inductively by

$$\delta^r f(x) = \delta^{r-1} f\left(x + \tfrac{1}{2}\right) - \delta^{r-1} f\left(x - \tfrac{1}{2}\right). \tag{9}$$

Denoting by $f(x_0 + k) = f_k$ the function value $k$ steps from an arbitrary initial argument $x_0$, the following closed-form expression results:

$$\delta^r f_k = \sum_{j=0}^{r} (-1)^j \binom{r}{j} f_{k+r/2-j}. \tag{10}$$

One may regard $\delta^r$ as a linear operator. It has the exponential property $\delta^m(\delta^n) = \delta^{m+n}$. Most importantly, it is a reductive operator; that is, it reduces the degree of any polynomial. In particular, the $r$th central difference operator annihilates a polynomial of degree $(r-1)$.

We can now examine the PCC expressions using these ideas. For example, from (10) the $m$th output signal (5) is seen to be

$$z_{m,i} = (-1)^{k-1} \sum_{n=0}^{\lfloor N/k \rfloor -1} = d_{n,i}\, \delta^{k-1} c_{nk-m+(k-1)/2} \tag{11}$$

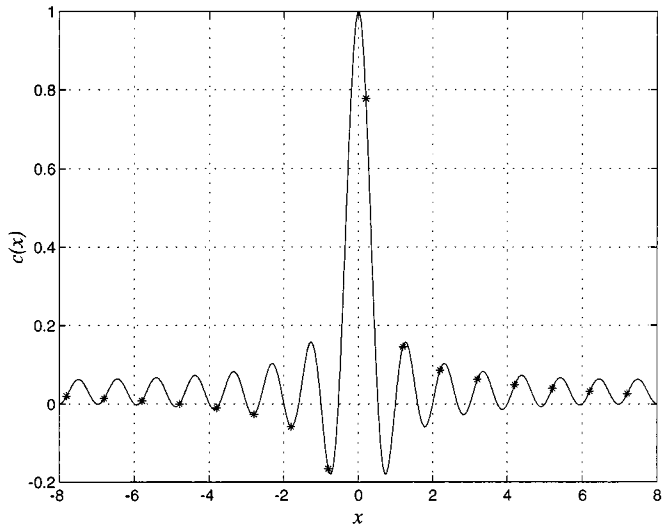
and the weighted combination (6) of received signals used to decode the $p$th piece of data is

$$\hat{d}_{p,i} = (-1)^{k-1} \delta^{k-1} z_{kp+(k-1)/2,i}. \tag{12}$$

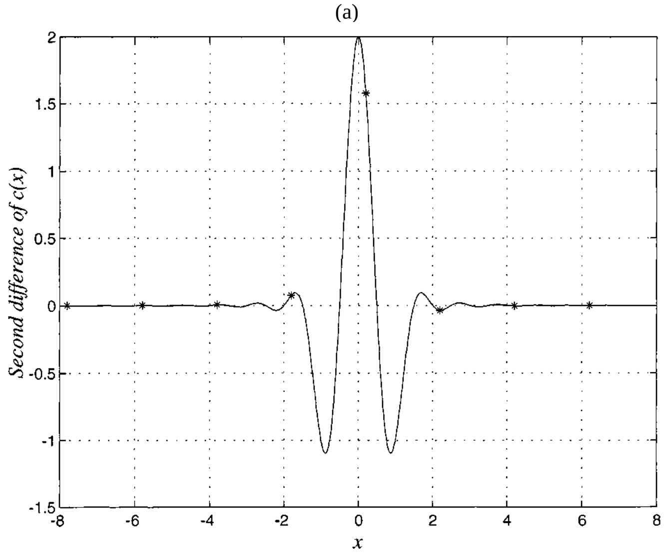
Care must be taken when these expressions are combined because $z_{m,i}$ is constructed from the $c_{nk-m}$. The operator acting on the first subscript of $z_{m,i}$, actually acts on $c_{nk-m}$ as $(-1)^{k-1}\delta^{k-1}$. Making use of the linearity and exponential properties of the operators, the $p$th complex value is decoded as

$$\hat{d}_{p,i} = (-1)^{k-1} \sum_{n=0}^{\lfloor N/k \rfloor -1} = d_{n,i} \delta^{2k-2} c_{k(n-p)}. \tag{13}$$

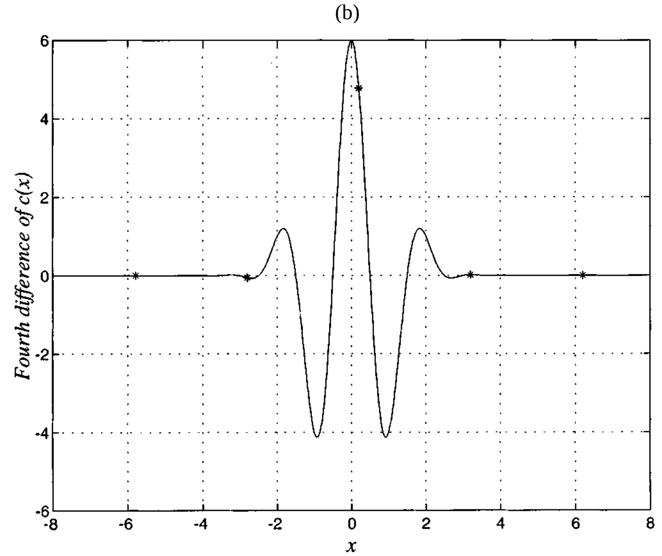
This is indeed (7) expressed in terms of the central difference (10).




(a)




(b)




(c)

Fig. 2. Plots of (17) for $N = 16$ and $k = 1$ (no PCC) and $k = 2, 3$ with $\Delta fT = 0.2$. The sampling effectively provided by PCC (13) is indicated (*).

## IV. Reduced ICI

By arriving at (13), we have demonstrated that for an OFDM system with PCC the coefficients in (1) are replaced by the (signed) $(2k-2)$th difference of a function on $\mathbb{R}$

$$c(x) = \frac{1}{N}\frac{\sin(\pi x)}{\sin(\pi x/N)} \exp\ (j\pi(N-1)x/N) \tag{14}$$

sampled according to

$$c_n = c(n + \Delta fT). \tag{15}$$

ICI is due to terms in (13) for which $n \neq p$.

To give a general expression for the finite differences of $c(x)$, we return to the original expression as a sum, rather than the closed form given in (2). It is readily established that

$$\delta^r e^{\alpha x+\beta} = (e^{\alpha/2} - e^{-\alpha/2})^r e^{\alpha x+\beta}. \tag{16}$$

Applying this we obtain

$$\begin{aligned}(-1)^{k-1}\delta^{2k-2}c(x) &= (-1)^{k-1}\frac{1}{N}\sum_{\ell=0}^{N-1}\delta^{2k-2}\exp\left(\frac{j2\pi\ell x}{N}\right)\\ &= \frac{4^{k-1}}{N}\sum_{\ell=0}^{N-1}\left(\sin\frac{\pi\ell}{N}\right)^{2k-2}\exp\left(\frac{j2\pi\ell x}{N}\right).\end{aligned} \tag{17}$$

Since $c(x)$ is trigonometric rather than polynomial in structure, $\delta^{2k-2}c_{k(n-p)}$ will not actually vanish for any value of $k$. (Of course, it should *not* vanish for $n = p$, since this corresponds to the wanted decoded signal.) Neither does the sum (17) have a closed form about which we could draw general conclusions. However, as was pointed out in [2], and is demonstrated in Fig. 2(a), the function $c(x)$ peaks about $x = 0$ and varies more slowly elsewhere. Fig. 2(b) and (c) shows that the effect of taking finite differences is to enhance the peak (the wanted signal, in practice normalized again to 1) relative to the rest of the graph. Due to the reductive nature of the finite-difference operators, PCC causes annihilation of polynomial variation up to order $(2k-3)$ in the slowly varying regions of the graph (the ICI). Indeed, as (11) makes clear, applying PCC only at the transmitter would give polynomial correction of order $(k-2)$. Fig. 2(a)–(c) also provide, with the sampling points marked, signal-to-ICI plots; the effective sampling (13) is seen to further the smoothing provided by the difference procedure. In practice, $\Delta fT$ is small, $N$ is large, and data transmission is improved even for relatively small values of $k$.

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## References

[1] J. Armstrong, "Improved data transmission," Australian patent application PO8271, 1997.

[2] ——, "Analysis of new and existing methods of reducing intercarrier interference due to carrier frequency offset in OFDM," *IEEE Trans. Commun.*, vol. 45, pp. 365–369, 1998.

[3] ——, "Polynomial cancellation coding with overlapping symbol periods," *IEEE J. Select. Areas Commun.*, 1999, submitted for publication.

[4] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. New York, NY: Dover , 1965.

[5] F. B. Hildebrand, *Introduction to Numerical Analysis*. New York, NY: McGraw-Hill, 1974.