Math 3331 – ODE's

Last class – Reduction of order

if $a(x)y'' + b(x)y' + c(x)y = 0$

given 1 sol^n y_1, look for 2^{nd} $y_2 = uy_1$

(we will use this in a bit)

Also, we will need ~~Euler's~~ Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Today – Find the general sol^n of

$$ay'' + by' + cy = 0 \quad (*) \qquad a, b, c \text{ constant}, \; a \neq 0$$

1st order

$$y' + py = 0 \quad (\text{sep})$$

$$\frac{dy}{y} = -p\,dx \qquad \ln|y| = -px + \ln c$$

$$y = c\,e^{-px}$$

so the sol^n looks like $y = e^{mx}$ m const so we try and find sol^{n}s to (*) like this. So

$$y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2 e^{mx}$$

sub $ay'' + by' + cy = 0$

$$am^2 e^{mx} + bme^{mx} + c \cdot e^{mx} = 0$$

$$am^2 + bm + c = 0 \quad (\text{since } e^{mx} \neq 0)$$

called characteristic eq^n (CE)

<u>Quadratic Formula</u>

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

<u>Cases</u>

(i) $b^2 - 4ac > 0$ real dist. roots
(ii) $b^2 - 4ac = 0$ real repeated
(iii) $b^2 - 4ac < 0$ complex.

Consider by examples

<u>Ex 1</u> $y'' - 3y' + 2y = 0$ $\quad m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0 \quad m = 1, 2$

$y_1 = e^x$
$y_2 = e^{2x}$ G.S. $\boxed{y = c_1 e^x + c_2 e^{2x}}$

Ex 2 $y'' - 4y' + 4y = 0$

CE $m^2 - 4m + 4 = 0$ $(m-2)^2 = 0$ $m = 2, 2$

1st sol^n $y = e^{2x}$

for second linearly indep sol^n

$y = e^{2x}u$, $y' = e^{2x}u' + 2e^{2x}u$

$y'' = e^{2x}u'' + 4e^{2x}u' + 4e^{2x}u$

sub $y'' - 4y' + 4y = 0$

$\cancel{e^{2x}}u'' + 4\cancel{e^{2x}}u' + 4\cancel{e^{2x}}u - 4(\cancel{e^{2x}}u' + 2\cancel{e^{2x}}u) + 4\cancel{e^{2x}}u = 0$

$u'' + \cancel{4u'} + \cancel{4u} - \cancel{4u'} - \cancel{8u} + \cancel{4u} = 0$

$u'' = 0$ $u = c_1x + c_2$

$y = (c_1x + c_2)e^{2x}$

$= c_1xe^{2x} + c_2e^{2x}$

$y_1 = e^{2x}$, $y_2 = xe^{2x}$ } so this is the general sol^n.

Ex3 $y'' - 2y' + 5 = 0$

CE $m^2 - 2m + 5 = 0$ $\quad m = \frac{2 \pm \sqrt{4 - 4(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$y_1 = e^{(1-2i)x}$, $y_2 = e^{(1+2i)x}$

G.S. $y = k_1 e^{x} \cdot e^{-2ix} + k_2 e^{x} e^{2ix}$ ← use Euler's Formula

$= k_1 e^{x}(\cos 2x - i \sin 2x)$

$k_2 e^{x}(\cos 2x + i \sin 2x)$

$= (k_1 + k_2) e^{x} \cos 2x + (-k_1 + k_2) i e^{x} \sin 2x$

let $k_1 + k_2 = c_1$, $(-k_1 + k_2) i = c_2$

so G.S. $y = c_1 e^{x} \cos 2x + c_2 e^{x} \sin 2x$

So. what about in general?

In general

if $ay'' + by' + cy = 0$

then if $y = e^{mx}$

CE $am^2 + bm + c = 0$

<u>Case 1</u> $m = r_1, r_2$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

<u>Case 2</u> $m = r, r$

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

(2nd lin. indep sol^n is $y = xe^{rx}$)

<u>Case 3</u> $m = \alpha \pm \beta i$

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x.$$

How to find the c_1, c_2

Given $y(x_0) = y_0$, $y(x_1) = y_1$

so initial cond