Trig Sub. 2

we continue our discussion with trig sub but now we have definite integrals i.e. w/ limits. This table will be helpful

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

so if we ask: for what θ is $\cos\theta = 1/2$

go along $\cos\theta$ line $\Rightarrow \frac{1}{2}$ then go up so

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \pi/3$$

similarly $\tan\theta = 1$ then $\theta = \pi/4$

These are the ones in ① |

ex 1 $\displaystyle\int_{\sqrt{2}}^{2} \frac{dx}{x^3\sqrt{x^2-1}}$

$x = \sec\theta$ $\qquad x^2-1 = \sec^2\theta - 1 = \tan^2\theta$

$dx = \sec\theta\tan\theta\,d\theta$ $\qquad \sqrt{x^2-1} = \tan\theta$

Limits $\quad \sec\theta = \sqrt{2} \quad \cos\theta = \frac{1}{\sqrt{2}} \quad \theta = \pi/4$

$\sec\theta = 2 \quad \cos\theta = \frac{1}{2} \quad \theta = \pi/3$

$$\int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta}{\sec^3\theta\tan\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_{\pi/4}^{\pi/3}$$

$$= \frac{\pi}{6} + \frac{\sin\frac{2\pi}{3}}{4} - \left(\frac{\pi}{8} + \frac{\sin\pi/2}{4}\right)$$

$$= \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}$$

ex 2 $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^2}}$

let $x = \tan\theta$ so $dx = \sec^2\theta\, d\theta$ & $\sqrt{1+x^2} = \sec\theta$

Limits $x = 0$ $\tan\theta = 0$ $\theta = 0$

$x = 1$ $\tan\theta = 1$ $\theta = \pi/4$

New $\int$ $\displaystyle\int_0^{\pi/4} \frac{\sec^2\theta}{\sec\theta}\, d\theta = \int_0^{\pi/4} \sec\theta\, d\theta$

$$= \ln|\sec\theta + \tan\theta| \Big|_0^{\pi/4}$$

$$= \ln\left|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right| - \underbrace{\ln|\sec 0 + \tan 0|}_{=0}$$

$$\sec\pi/4 = \frac{1}{\cos\pi/4} = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}$$

$\tan\pi/4 = 1$

$$= \ln|\sqrt{2} + 1|$$

Ex 3 $\int_0^1 x^3\sqrt{1-x^2}\,dx$

$x = \sin\theta$ $\quad \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

$dx = \cos\theta\,d\theta$

limits $\quad x=0 \quad \sin\theta = 0 \quad \theta = 0$

$x = 1 \quad \sin\theta = 1 \quad \theta = \pi/2$

$$\int_0^{\pi/2} \sin^3\theta \cdot \cos\theta \cdot \cos\theta\,d\theta = \int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$$

trig $\int$

$$\int_0^{\pi/2} \sin^2\theta\cos^2\theta\sin\theta\,d\theta$$ let $u = \cos\theta \quad du = -\sin\theta\,d\theta$

$\theta = 0 \quad u = \cos 0 = 1$

$\theta = \pi/2 \quad u = \cos\pi/2 = 0$

$$-\int_1^0 (1-u^2)u^2\,du$$

$$\int_0^1 (u^2-u^4)\,du = \frac{u^3}{3} - \frac{u^5}{5}\Big|_0^1 = \frac{1}{3} - \frac{1}{5}$$

$$= \frac{2}{15}$$

ex $\int \frac{dx}{\sqrt{4x-x^2}}$?

Here we will try & complete the square

$$4x - x^2 = -(x^2 - 4x)$$
$$= -(x^2 - 4x + 4 - 4)$$
$$= 4 - (x^2 - 4x + 4) = 4 - (x-2)^2$$

so $\int \frac{dx}{\sqrt{4-(x-2)^2}}$ let $x - 2 = 2\sin\theta$
$dx = 2\cos\theta\, d\theta$

$$4 - (x-2)^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$$

$$\int \frac{2\cos\theta\, d\theta}{2\cos\theta} = \int d\theta = \theta + C$$

$\sin\theta = \frac{x-2}{2}$ so $= \sin^{-1}\left(\frac{x-2}{2}\right) + C$