Ontario Math Circles
Second Annual ARML Team Selection Test
Solutions

1. Determine the unit's digit of $222 \times 3333 - 7777777^2$.
$\boxed{3}$ It suffice to compute $2 \times 3 - 7^2 = -43$. Therefore, the answer is 3.

2. Determine the solutions to the equation $x^2 - 2x + 6 = 0$.
$\boxed{1 \pm i\sqrt{5}}$ By the quadratic formula,

$$x = \frac{2 \pm \sqrt{2^2 - 4(6)}}{2} = 1 \pm \sqrt{-5} = 1 \pm i\sqrt{5}$$

3. Determine the area of the triangle bounded by the y-axis and the following two lines:

$$y = -x + 1$$
$$y = 2x - 2$$

$\boxed{\frac{3}{2}}$ The intersection of these two lines is at $(1, 0)$ and the y-intercepts are at $(0, 1)$ and $(0, -2)$. Therefore, the area is $\frac{1}{2}(1)(3) = \frac{3}{2}$.

4. Thinula can make a contest in 40 minutes. Bill can make a contest in 30 minutes. Eddy hires both of them to make one contest, in how many hours does it take for the job to be done.
$\boxed{\frac{2}{7}}$

$$\frac{1}{\frac{2}{3}} + \frac{1}{\frac{1}{2}} = \frac{1}{T}$$

Solving for $T = \frac{2}{7}$.

5. In terms of area, a regular heptagon with side length 1 is compared to a regular octagon with side length 1. Determine the area of the larger one.
$\boxed{2 + 2\sqrt{2}}$ The larger shape is obvious the octagon, which has area $2 + 2\sqrt{2}$.

6. Let $f(x) = x^4 - x^3 + ax + b$ with $f(1) = 4$ and $f(2) = 6$. Determine the ordered pair (a, b)?
$\boxed{(-6, 10)}$ The two given points yields the system of equations

$$\begin{cases} a + b = 4 \\ 2a + b = -2 \end{cases}$$

Solving this yields $a = -6$ and $b = 10$. Therefore, the answer is $(-6, 10)$.

7. Determine the smallest positive integer which is divisible by all one digit positive integers.
$\boxed{2520}$ The LCM of these nine numbers is

$$\text{lcm}(2, 3, 4, 5, 6, 7, 8, 9, 10) = \text{lcm}(8, 5, 7, 9) = 2520$$

8. Let P be a point inside a convex quadrilateral $ABCD$. Determine the average of the following four angles: $\angle APB, \angle PBC, \angle CPD, \angle DPA$.
$\boxed{90^\circ}$ The sum of the four angles is a constant, 360°. Therefore, the average is always 90°.

9. Determine the number of positive perfect square divisors of 10!.
$\boxed{30}$ The prime factorization of 10! is $2^8 3^4 5^2 7$. The number of positive even divisors is

$$\left(\left\lfloor \frac{8}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{4}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{2}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{1}{2} \right\rfloor + 1\right) = (5)(3)(2)(1) = 30$$

10. The permutations of ELDYD are listed in lexigraphical order. Determine the 26^{th} permutation in this list.
$\boxed{\text{EDDYL}}$ Starting with D, there are $4! = 24$ permutations. Therefore, the 25^{th} permutation is EDDLY and the desired permutation is EDDYL.

11. Consider all lattice points on the Cartesian plane that are of distance 5 from the origin. Determine the perimeter of this convex polygon.
$\boxed{8\sqrt{10}+4\sqrt{2}}$ In the first quadrant with the positive axises, the points are $(5,0)$, $(4,3)$, $(3,4)$, $(0,5)$. The total length of these 3 segments is $\sqrt{10}+\sqrt{2}+\sqrt{10} = 2\sqrt{10}+\sqrt{2}$. Therefore, the perimeter of the convex polygon is $8\sqrt{10}+4\sqrt{2}$.

12. How many three digit numbers contain the digit "9" at least once?
$\boxed{252}$ The number of three digit numbers is 900. The number of three digit numbers that does not contain 9 is $(8)(9)(9) = 648$. Therefore, the answer is $900 - 648 = 252$.

13. Let a and b be the solutions to $x^2 - 2x + 6 = 0$. Determine the value of $a^6 + b^6$.
$\boxed{352}$

$$
\begin{aligned}
&a^6 + b^6 \\
&= \left(a^3+b^3\right)^2 - 2\,(ab)^3 \\
&= \left((a+b)(a^2-ab+b^2)\right)^2 - 2\,(ab)^3 \\
&= \left((a+b)((a+b)^2-3ab)\right)^2 - 2\,(ab)^3 \\
&= \left((2)((2)^2-3(6))\right)^2 - 2\,(6)^3 \\
&=352
\end{aligned}
$$

14. Determine all positive integer solutions (x,y,z) to the system of equations

$$\begin{cases} xy + yz = 63 \\ xz + yz = 23 \end{cases}$$

$\boxed{(2,21,1),(20,3,1)}$ The second equation implies that either $z = 1$ or $z = 23$. If $z = 23$ then $x + y = 1$. This is not possible because x and y must be positive integers. If $z = 1$ then

$$\begin{cases} xy + yz = 63 \\ x + y = 23 \end{cases}$$

Substituting the second equation into the first yields

$$x^2 - 22x + 40 = 0$$

This has solutions $x = 2$ and $x = 20$. The corresponding y value are 21 and 3, respectively. Therefore, the possible integer solutions are $(2,21,1)$ and $(20,3,1)$.

15. Determine the sum of all positive real numbers x for which $\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$.
$\boxed{8 + \frac{1}{2}\sqrt[3]{2}}$

$$
\begin{aligned}
\frac{\log_2 x}{2} - \frac{4}{\log_2 x} &= \frac{7}{6} - \frac{3}{\log_2 x} \\
\left(\log_2 x + \frac{2}{3}\right)(\log_2 x - 3) &= 0 \\
x &= 2^{-\frac{2}{3}}, 8 \\
&= \frac{\sqrt[3]{2}}{2}, 8
\end{aligned}
$$

16. Determine the number of real solutions to $\log_4 x = 2\sin x$.
$\boxed{5}$ For $x > 16$, there are no solutions. A reasonably sketch will show that there are 5 points of intersection.

17. The distance between the centers of two circles is 15. One has radius 4 and the other has radius 5. What is the length of their common internal tangent?
$\boxed{12}$ Consider the following diagram: . By similar triangles the answer is 12.

18. Let $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Compute the value of $\sum_{n=1}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}}$.
$\boxed{2}$ Note $F_n = F_{n+1} - F_{n-1}$ so

$$\sum_{n=1}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = \sum_{n=1}^{\infty} \frac{F_{n+1} - F_{n-1}}{F_{n-1}F_{n+1}} = \sum_{n=1}^{\infty} \left(\frac{1}{F_{n-1}} - \frac{1}{F_{n+1}} \right)$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = \sum_{n=1}^{\infty} \frac{1}{F_{n-1}} - \sum_{n=1}^{\infty} \frac{1}{F_{n+1}} = F_0 + F_1 + \sum_{n=2}^{\infty} \frac{1}{F_n} - \sum_{n=2}^{\infty} \frac{1}{F_n} = 2$$

19. Let $f(x) = \frac{x}{\sqrt{1+x^2}}$ and define $f^{(n)}(x) = \underbrace{f(f(f \cdots f}_{n}(x)))$. Determine $f^{(99)}(1)$.
$\boxed{\dfrac{1}{10}}$ By induction, it can be shown that $f^{(n)}(x) = \frac{x}{\sqrt{nx^2+1}}$. Therefore, $f^{(99)}(1) = \frac{1}{\sqrt{99+1}} = \frac{1}{10}$.

20. If the sum of all positive divisors of 30^{120} which are multiples of 30^{118} is $30^{118}N$, compute the value of N.
$\boxed{2821}$ Note each such divisor is of the form $2^a 3^b 5^c$ where $118 \leq a, b, c \leq 120$. The desired sum is then

$$\begin{aligned} &\left(2^{118} + 2^{119} + 2^{120}\right)\left(3^{118} + 3^{119} + 3^{120}\right)\left(5^{118} + 5^{119} + 5^{120}\right) \\ &= 30^{118}\left(1 + 2 + 2^2\right)\left(1 + 3 + 3^2\right)\left(1 + 5 + 5^2\right) \\ &= 30^{118}(7)(13)(31) \end{aligned}$$

Therefore, $N = (7)(13)(31) = 2821$.

21. Let a and b be positive integers such that $a > b + 1 > 2$. If $\binom{13}{5} + \binom{13}{6} = \binom{a}{b}$, compute the maximum possible value of $a + b$.
$\boxed{154}$ Note $\binom{13}{5} + \binom{13}{6} = \binom{14}{6} = 77 \times 39 = \binom{78}{2} = \binom{78}{76}$. To prove that the pair $(a, b) = (78, 76)$ is the desired maximum, note that if $a \leq 78$, there is no other possible maximum. However, if $a > 78$, since $b > 1$, it follows that $\binom{a}{b} > \binom{79}{2} = 79 \times 39 > \binom{14}{6}$, a contradiction. Hence, our desired values are $a = 78$ and $b = 76$ so $a + b = 154$.

22. If $\sin x + \cos x = \frac{1}{5}$ and $0 \leq x < \pi$. Determine $\tan x$.
$\boxed{-\dfrac{4}{3}}$ Squaring the given quantity yields

$$1 + 2\sin x \cos x = \frac{1}{25}$$

Hence, $\sin x$ and $\cos x$ are the roots of $25t^2 - 5t - 12 = 0$. Therefore, $\sin x = \frac{4}{5}$ and $\cos x = -\frac{3}{5}$. Therefore, $\tan x = -\frac{4}{3}$.

23. A fair die is rolled repeatedly. Given that 6 is obtained for the first time on the second roll, compute the expected number of rolls to obtain a 5 for the first time.

$\boxed{\frac{33}{5}}$ The probability that 5 appears on the first roll is $\frac{1}{5}$. The probability that 5 appears for the first time on the n^{th} roll where $n > 2$ is $\frac{4}{5} \cdot \left(\frac{5}{6}\right)^{n-3} \cdot \frac{1}{6}$. Therefore, the answer is

$$\frac{1}{5} + \sum_{n=3}^{\infty} n \cdot \frac{4}{5} \cdot \left(\frac{5}{6}\right)^{n-3} \cdot \frac{1}{6} = \frac{33}{5}$$

24. A point P is inside unit square $ABCD$. Let a, b, c, d be the area of $\triangle APD$, $\triangle DCP$, $\triangle BCP$, and $\triangle ABP$, respectively. Determine the length of the curve of points of P such that

$$\begin{aligned} ax + by &= 2017 \\ cx + dy &= 2018 \end{aligned}$$

does not have a distinct solution.
$\boxed{\sqrt{2}}$ It suffice to let a, b, c, d to represent the distance from P to AD, CD, BC, AB, respectively. Thus, $c = 1 - a$ and $b = 1 - d$. For a system of equations to not have a distinct solution, $ad - bc = 0$. This implies that $ad - (1 - d)(1 - a) = 0$, which is $a + d = 1$. This represents the points on the diagonal BD, which has length $\sqrt{2}$.

25. Let $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Determine the value of $\sum_{n=0}^{\infty} \frac{F_n}{4^n}$.
$\boxed{\frac{4}{11}}$ Let $f(x) = \sum_{n=0}^{\infty} F_n x^n$ then

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} F_n x^n \\ &= \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2})\, x^n + x \\ &= x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2} + x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} + x \\ &= x^2 f(x) + x f(x) + x \\ &= \frac{x}{1 - x - x^2} \end{aligned}$$

Therefore, $f\left(\frac{1}{4}\right) = \frac{\frac{1}{4}}{1 - \frac{1}{4} - \left(\frac{1}{4}\right)^2} = \frac{4}{11}$

26. Evaluate $\sqrt[3]{20 + 14\sqrt{2}} + \sqrt[3]{20 - 14\sqrt{2}}$.
$\boxed{4}$

$$\begin{aligned} &\sqrt[3]{20 + 14\sqrt{2}} + \sqrt[3]{20 - 14\sqrt{2}} \\ =&\sqrt[3]{\left(2 + \sqrt{2}\right)^3} + \sqrt[3]{\left(2 - \sqrt{2}\right)^3} \\ =&2 + \sqrt{2} + 2 - \sqrt{2} \\ =&4 \end{aligned}$$

27. In $\triangle ABC$, $\angle A = 90^\circ$, $AC = 1$, $AB = 5$. Point D lies on ray $\vec{AC}$ such that $\angle DBC = 4\angle CBA$. Compute AD.
$\boxed{\frac{719}{95}}$ Since $(5 + i)^5 = 1900 + 2876i$ then $AD = 5 \times \frac{2876}{1900} = \frac{719}{95}$.

28. Let x be the smallest real value such that $\lfloor x^2 \rfloor - \lfloor x \rfloor^2 = 2017$. Determine $\lfloor x \rfloor$.
$\boxed{1009}$ Let $x = n + a$ where n is an integer and $0 \le a < 1$. Substituting this into the equation yields

$$\lfloor 2na + a^2 \rfloor = 2017$$

To find the smallest n, a needs to be as large as possible. If $a = 1$ then $n = \frac{2017-1}{2} = 1008$ will suffice. However, since a cannot be 1 then $1009 < x < 1010$. Therefore, $\lfloor x \rfloor = 1009$.

29. Let $f(n) = n(n+1)(n+2)(n+3)(n+4)$. If $T = \sum_{n=1}^{2016} f(n)$, and S is the sum of all factors of T, compute the value when S is divided by 50.
$\boxed{0}$ Note that $f(n) = 120\binom{n+4}{5}$. We then seek to compute $120\left(\binom{5}{5} + \binom{6}{5} + \cdots + \binom{2020}{5}\right)$. By the hockey-stick formula, this is equal to $120\binom{2021}{6}$, which has a prime factorization of

$$2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 43 \cdot 47 \cdot 101 \cdot 673 \cdot 1009 \cdot 2017$$

This means that

$$S = (1+2+2^2+2^3+2^4)(1+3)(1+5)(1+7)(1+43)(1+47)(1+101)(1+673)(1+1009)(1+2017) \equiv 0 \bmod 50$$

there both 25 and 2 divide the product. Therefore, the remainder when S is divided by 50 is 0.

30. Determine all ordered triples (x, y, z) of nonnegative real numbers such that $xy + yz + xz = 12$ and $xyz = x + y + z + 2$.
$\boxed{(2,2,2)}$ Let $t = \sqrt[3]{xyz}$. Using AM-GM on the first equation implies that $12 \ge 3t^2$ or $t \le 2$. By AM-GM on the second equation, we have $t^3 \ge 3t + 2$. This re-arranges to $0 \le t^3 - 3t - 2 = (t-2)(t+1)^2$. Since $t \ge 0$ we know that this inequality implies $t \ge 2$. Therefore, $2 \le t \le 2$ so $t = 2$. This means $xyz = 8$, $xy + yz + xz = 12$ and $x + y + z = 6$. Note that x, y, z are the three roots of the polynomial $0 = r^3 - 6r^2 + 12r - 8 = (r-2)^3$ which implies $(x, y, z) = (2, 2, 2)$ is the only solution.

31. Let $f(x) = x^3 - 2018x + 1$ and $g(x) = \frac{x-1}{x^2}$. Compute the number of, not necessarily distinct, real solutions to $f(g(x)) = 0$.
$\boxed{4}$ It can be shown that $f(-45) < 0 < f(0)$, $f(0) > 0 > f\left(\frac{1}{4}\right)$ and $f(1) < 0 < f(45)$. Therefore the roots α, β, γ of $f(x)$ must satisfy $-45 < \alpha < 0$, $0 < \beta < \frac{1}{4}$ and $1 < \gamma < 45$. Note if $g(x) = a$, then $ax^2 - x + 1 = 0$ which has discriminant $1 - 4a$. Therefore, $g(x) = a$ will have two real solutions if $0 \ne a \le \frac{1}{4}$ and zero solutions if $a > \frac{1}{4}$. Since $0 \ne \alpha, \beta < \frac{1}{4}$ and $\gamma > \frac{1}{4}$, there will be $2 \times 2 = 4$ real solutions to $f(g(x)) = 0$.

32. Let n be a natural number and let $S(n)$ denote the sum of the digits of n in its base 10 representation. Determine all natural numbers n such that $n^3 = 8S(n)^3 + 6nS(n) + 1$.
$\boxed{17}$ We can rewrite the given condition as $n^3 + (-2S(n))^3 + (-1)^3 = 3n(-2S(n))(-1)$. Note that this is in the form $x^3 + y^3 + z^3 = 3xyz$ which happens if and only if $x + y + z = 0$. Therefore, $n = 2S(n) + 1$. It is easy to see that the number of digits in n cannot be more than 2. Since $n = 2S(n) + 1 \le 18k + 1$ where$k$ is the number of digits of$n$. Then, if n has at least3 digits, $n > 33k$ so that yields $33k \le 18k + 1$, a contradiction. Therefore, n has at most 2 digits and it clearly does not have 1 digit since $S(n) = n$ in that case. Thus, n must have 2 digits. Let $n = 10a + b$. Then, $S(n) = a + b$ so we have $10a + b = 2a + 2b + 1$ meaning $8a = b + 1$. Since$a$ and b are digits, this implies $a = 1$ and $b = 7$ which implies $n = 17$ is the only solution.

33. A portion of Thinula's test paper is cut off and he can only see the first three terms of $f(x) = x^{10} - 10x^9 + 45x^8$. Given that all the roots of the polynomial are real, determine the product of the roots.
$\boxed{1}$ Note$\sum_{i=1}^{10} r_i^2 = 10^2 - 2(45) = 10$. Therefore, $10\sum_{i=1}^{10} r_i^2 = 100$ which implies $\frac{r_1}{1} = \frac{r_2}{1} = \ldots = \frac{r_{10}}{1}$ by Cauchy-Schwarz. Therefore, $(r_1, r_2, \ldots, r_{10}) = (1, 1, \ldots, 1)$ is the only possible ordered 10-tuple of roots so the desired product is 1.

34. In $\triangle ABC$, AD is an angle bisector, D is on BC. A circle centred at O is inscribed in $\triangle ABD$. Let E be on AB such that OE is perpendicular to AB. If $BE = 2$, $BD = 3$, $AE = 4$, compute AC.
$\boxed{\frac{50}{9}}$ Note that E is the point of tangency with the circle and AB. By equal tangents, $AD = 5$. Let $DC = a$ and $AC = b$. By angle bisector theorem, $b = 2a$. By Stewart's Theorem,

$$6^2a + 3b^2 = 5^2(a+3) + 3a(a+3)$$

Solving this yields $a = \frac{25}{9}$ and, thus, $b = \frac{50}{9}$. Therefore, $AC = \frac{50}{9}$.

35. Let $f(x)$ be a degree 8 polynomial such that $f(k) = 2^k$ for $k = 0, 1, 2, \ldots, 8$, compute the value of $f(9)$.
$\boxed{511}$ Since it is in an integer at all integer inputs, $f(x)$ can be written as a linear combination of the binomial coefficients $\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{n}$. Therefore, $f(x) = \binom{x}{0} + \binom{x}{1} + \ldots + \binom{x}{8}$ so $f(9) = 2^9 - \binom{9}{9} = 2^9 - 1 = 511$.

36. Thinula has some coins $C_1, C_2, \ldots, C_n$. Each coin is biased so that the probability of getting heads on coin C_k is $\frac{1}{k+2}$ for all $1 \le k \le n$. When Thinula tosses all the coins, the probability of getting an odd number of heads is $\frac{2015}{4032}$. How many coins does Thinula have?
$\boxed{62}$ Consider the polynomial $f(X) = \prod_{k=1}^{n} \left(\left(1 - \frac{1}{k+2}\right) + \frac{1}{k+2}X\right)$. Note the coefficient of X^m represents the probability of flipping exactly m heads. We wish to compute the sum $S = a_1 + a_3 + \ldots$. Note that

$$2S = f(1) + f(-1) = 1 - \prod_{k=1}^{n} \left(\frac{k}{k+2}\right) = 1 - \frac{2}{(n+1)(n+2)} = \frac{2015}{2016}$$

Therefore, $\dfrac{2}{(n+1)(n+2)} = \dfrac{1}{2016}$ so $(n+1)(n+2) = 4032 = 63(64)$ so $n = 62$.

37. Define the sequence $a_1, a_2, \ldots$ to be a sequence such that $a_1 = 1$ and for $n \ge 1$,

$$16a_{n+1} = 1 + 4a_n + \sqrt{1 + 24a_n}$$

Evaluate $\sum_{n=1}^{\infty} \left(a_n - \frac{1}{3}\right)$.
$\boxed{\frac{11}{9}}$ Let $b_n = \sqrt{1 + 24a_n}$ then $b_1 = 5$

$$16\left(\frac{b_{n+1}^2 - 1}{24}\right) = 1 + 4\left(\frac{b_n^2 - 1}{6}\right) + b_n \implies (2b_{n+1})^2 = (b_n + 3)^2$$

Since $b_n > 0$ then

$$b_{n+1} = \frac{1}{2}b_n + \frac{3}{2} \implies b_n = 2^{2-n} + 3$$

Thus, $a_n = \frac{1}{3} + \frac{2}{3}\left(\frac{1}{4}\right)^n + \left(\frac{1}{2}\right)^n$. Therefore,

$$\sum_{n=1}^{\infty} \left(a_n - \frac{1}{3}\right) = \sum_{n=1}^{\infty} \left(\frac{2}{3}\left(\frac{1}{4}\right)^n + \left(\frac{1}{2}\right)^n\right) = \frac{11}{9}$$

38. Rhombus $ARML$ has its vertices on the graph of $y = 2\lfloor x \rfloor - x$. Given that the area of $ARML$ is 8, compute the least upper bound for $\tan A$.
$\boxed{\frac{100}{621}}$ This is the same question as 2017 ARML individual #10. You looked at the solution after the contest right?!?!?

39. Let N be the number of polynomials such that its coefficients belong to the set $\{1, 2, \ldots, 2018\}$ and $2(P(x)+1) = P(x+1) + P(x-1)$. Compute the remainder when N is divided by 1000.

$\boxed{324}$ Let $Q(x) = P(x) - P(x-1) - 2x$. Rearranging the original condition yields

$$P(x+1) - P(x) = P(x) - P(x-1) + 2$$

Subtracting $2x + 2$ from both sides yield

$$P(x+1) - P(x) - (2x+2) = P(x) - P(x-1) - 2x$$

so $Q(x+1) = Q(x)$ for every x. Since Q is a polynomial, it must be constant so

$$P(x) = P(x-1) + 2x + k$$

for some $k \in \mathbb{R}$. Next, let $T(x) = P(x) - x^2 - (k+1)x$ so $P(x) = T(x) + x^2 + (k+1)x$ and $P(x-1) = T(x-1) + (x-1)^2 + (k+1)(x-1)$. This means

$$P(x) - P(x-1) = 2x + k = T(x) - T(x-1) + 2x + k$$

Therefore, $T(x) = T(x-1)$ so $T(x) = m$ for some $m \in \mathbb{R}$. Hence

$$P(x) = T(x) + x^2 + (k+1)x = x^2 + (k+1)x + m$$

Therefore, $P(x) = x^2 + ax + b$ where $a, b \in \{1, 2, \cdots, 2018\}$. Hence,

$$N \equiv 2018^2 \equiv 18^2 \equiv 324 \pmod{1000}$$

40. Let $d(P, MN)$ denote the smallest distance between point P and line segment MN. Let G be the centroid of $\triangle ABC$ and let X be the point in the plane such that $\frac{d(X,AC)}{AC} = \frac{d(X,AB)}{AB} = \frac{d(X,BC)}{BC}$. Let G_a, G_b, G_c be the feet of the perpendiculars from G to BC, AC and AB respectively. Similarly, let X_a, X_b, X_c be the feet of the perpendiculars from X to BC, AC and AB respectively. If O_1 and O_2 are the circumcenters of $\triangle G_aG_bG_c$ and $\triangle X_aX_bX_c$ respectively, compute the value of $\frac{O_1G+O_2X}{2XG}$.

$\boxed{\frac{1}{2}}$ Notice X is the Lemoine point so X and G are isogonal conjugates. It follows from this that $O_1 = O_2 = M$, the midpoint of XG. Therefore, the desired ratio is $\frac{1}{2}$.