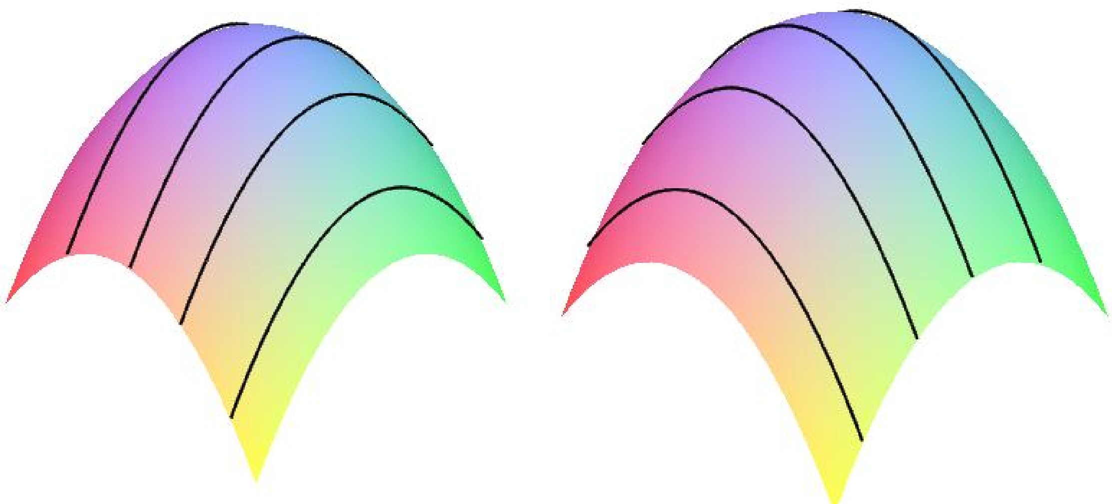
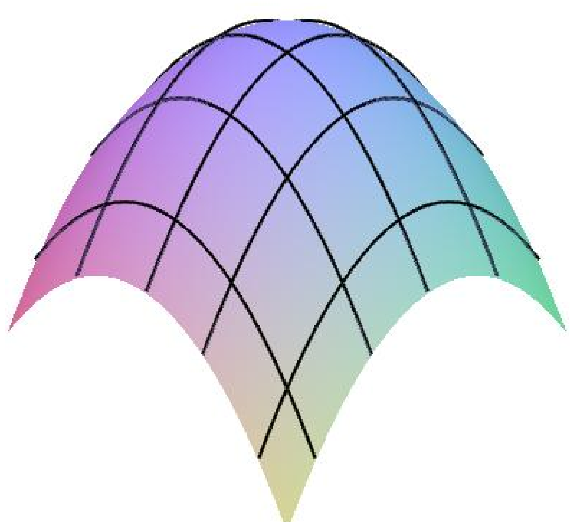
Calculus 3 - Parametric Surfaces

When we first introduced surfaces, say $z = f(x, y)$, one way to draw them is to fix y to a certain value, say $y = c$, and then sketch the space curve $z = f(x, c)$. As we vary c, we get different spacecurves and together, they give a graph of the surface. Similarly, fix $x = k$ and sketch the space curve $z = f(k, y)$.



These two together sketches the entire surface



When we first introduced vector functions

$$\vec{r}(t) = \left\langle f(t), g(t), h(t) \right\rangle \tag{1}$$

we found that the tip of the vector touched a space curve given by

$$x = f(t), \quad y = g(t), \quad z = h(t).$$

Parametric Surfaces

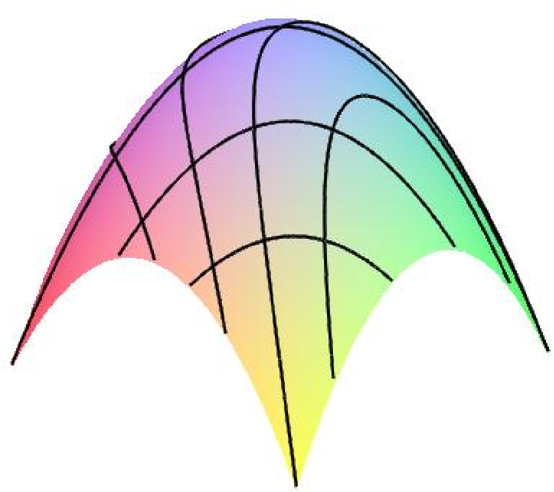
Let x, y and z be functions of u and v that are continuous in some domain D. The set of points (x, y, z) given by

$$\vec{r}(u, v) = \left\langle x(u, v), y(u, v), z(u, v) \right\rangle \tag{2}$$

is called a parametric surface and

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v) \tag{3}$$

are the parametric equations of the surface. In this figure, we fix v and vary



u and then fix u and vary v.

Example 1.

Consider the parametric surface

$$\vec{r}(u,v) = \left\langle u,\ v,\ \sqrt{u^2+v^2} \right\rangle \tag{4}$$

If we identify that

$$x = u, \quad y = v, \quad z = \sqrt{u^2+v^2}, \tag{5}$$

then we see that

$$z = \sqrt{x^2+y^2} \tag{6}$$

the equation of the cone (top half). However, consider

$$\vec{r}(u,v) = \left\langle v\cos u,\ v\sin u,\ v \right\rangle. \tag{7}$$

If we identify that

$$x = v\cos u, \quad y = v\sin u, \quad z = v \tag{8}$$

then eliminating u and v we get (6), the same cone. If we specify intervals, say

$$0 \le u \le 2\pi, \quad 0 \le v \le 1 \tag{9}$$

we will get a specific part of the cone (of course, letting u go beyond 2π would just repeat what we already have).

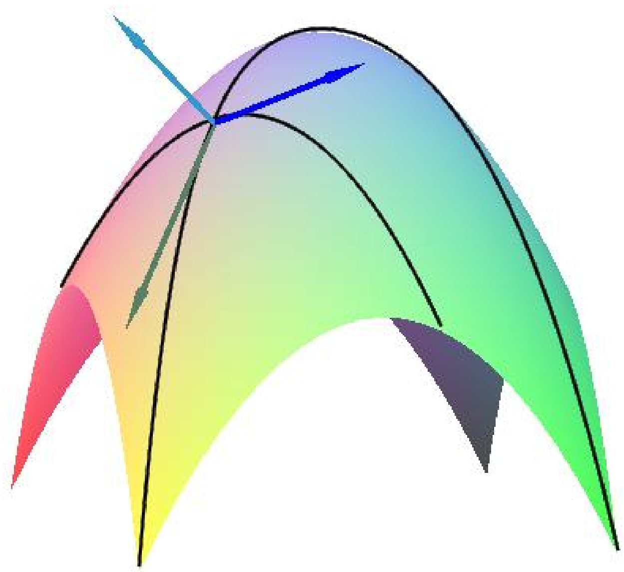
Normal Vectors and Tangent Planes

Given the parametric surface

$$\vec{r}(u,v) = \Big\langle x(u,v),\, y(u,v),\, z(u,v) \Big\rangle \tag{10}$$

if we fix v (or u) we create a spacecurve. To that curve we can create a tangent vector. This is obtained by differentiating with respect to the varying variable. Thus, we have two tangent vectors (green and blue)

$$\vec{r}_u = \Big\langle x_u,\, y_u,\, z_u \Big\rangle, \quad \vec{r}_v = \Big\langle x_v,\, y_v,\, z_v \Big\rangle \tag{11}$$



The normal vector $\vec{N}$ is given by (evaluated at some (u_0, v_0))

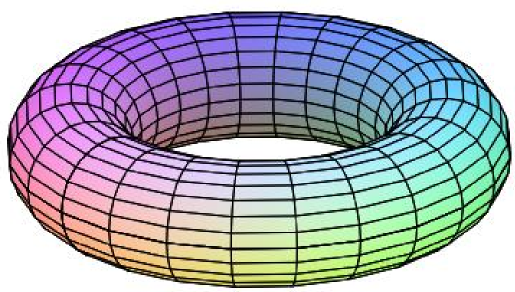
$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \tag{12}$$

Example 2.

Find the equation of the tangent plane to the parametric surface

$$\vec{r}(u,v) = \Big\langle \Big(3+\cos(v)\Big)\cos(u), \Big(3+\cos(v)\Big)\sin(u), \sin(v)\Big\rangle \\ 0 \le u \le 2\pi, \quad 0 \le v \le 2\pi \tag{13}$$

at the point $(3, 0, 1)$. The sketch is below.



Soln.

We will first find the corresponding u and v. So

$$\Big(3+\cos(v)\Big)\cos(u) = 3, \quad \Big(3+\cos(v)\Big)\sin(u) = 0, \quad \sin(v) = 1. \tag{14}$$

Solving gives

$$u = 0, \quad v = \pi/2. \tag{15}$$

Next we find derivatives so

$$\vec{r}_u = \Big\langle -\Big(3+\cos(v)\Big)\sin(u), \Big(3+\cos(v)\Big)\cos(u), 0\Big\rangle \\ \vec{r}_v = \Big\langle -\sin(v)\cos(u), -\sin(v)\sin(u), \cos(v)\Big\rangle \tag{16}$$

Next we evaluate these at the point given in (15) so

$$\vec{r}_u = \left\langle 0, 3, 0 \right\rangle, \quad \vec{r}_v = \left\langle -1, 0, 0 \right\rangle \tag{17}$$

so

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 0 \\ -1 & 0 & 0 \end{vmatrix} = \langle 0, 0, 3 \rangle \tag{18}$$

The equation of the tangent plane is therefore

$$0(x-3) + 0(y-0) + 3(z-1) = 0 \tag{19}$$

so simply $z = 1$.

Surface Area

We would like to find the surface area of a given the parametric surface

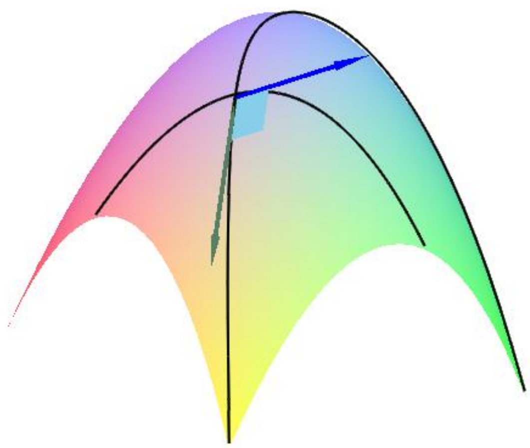
$$\vec{r}(u, v) = \left\langle x(u, v), y(u, v), z(u, v) \right\rangle \tag{20}$$

If we scale each tangent vector by a small amount say du and dv then we have

$$\vec{r}_u du, \quad \vec{r}_v dv, \tag{21}$$

the area of a small parallelogram is given by

$$dS = \left\| \vec{r}_u \times \vec{r}_v \right\| du dv \tag{22}$$



the the required surface is (on adding the small areas)

$$SA = \iint\limits_{R_{uv}} \|\vec{r}_u \times \vec{r}_v\| \, dudv \tag{23}$$

where R_{uv} is some region in the uv plane which maps out the surface.

Example 3.

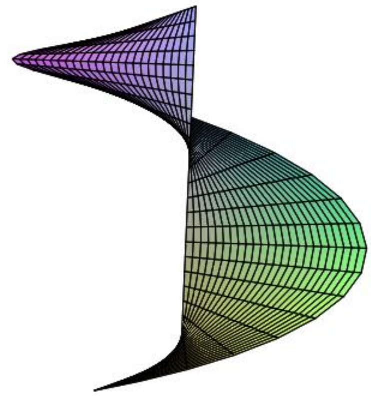
Find the surface area of the ramp function given by the parametric surface

$$\begin{gathered} \vec{r}(u,v) = \Big\langle u\cos v,\, u\sin v,\, v \Big\rangle \\ 0 \le u \le 1, \quad 0 \le v \le 2\pi \end{gathered} \tag{24}$$

Soln.

We first calculate derivatives

$$\begin{aligned} \vec{r}_u &= \Big\langle \cos v,\, \sin v,\, 0 \Big\rangle \\ \vec{r}_v &= \Big\langle -u\sin v,\, u\cos v,\, 1 \Big\rangle \end{aligned} \tag{25}$$



then cross them

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u \rangle \tag{26}$$

Then take the magnitude so

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2} \tag{27}$$

So, the surface area is

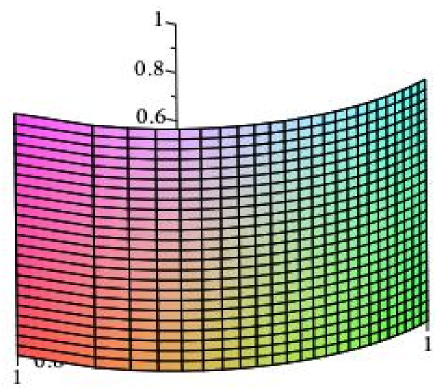
$$\begin{aligned} SA &= \int_0^1 \int_0^{2\pi} \sqrt{1+u^2} dv du \\ &= \int_0^1 \sqrt{1+u^2}\ v\Big|_0^{2\pi} du \\ &= 2\pi \int_0^1 \sqrt{1+u^2} du \\ &= 2\pi \cdot \frac{1}{2}\left(u\sqrt{1+u^2} + \ln\left|u+\sqrt{1+u^2}\right|\right)\Big|_0^1 \\ &= \left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right)\pi \end{aligned} \tag{28}$$

Surface Integrals

Here we will return to a problem we considered once before. Evaluate

$$\iint_S y dS. \tag{29}$$

where S is the surface of the cylinder $x^2 + y^2 = 1$ $(x, y \geq 0)$ for $0 \leq z \leq 1$.



Here we parametrize the surface by

$$\begin{aligned} \vec{r}(u,v) &= \Big\langle \cos u, \sin u, v \Big\rangle \\ 0 \leq u \leq \pi/2, &\quad 0 \leq v \leq 1 \end{aligned} \tag{30}$$

We first calculate derivatives

$$\begin{aligned} \vec{r}_u &= \Big\langle -\sin v, \cos v, 0 \Big\rangle \\ \vec{r}_v &= \Big\langle 0, 0, 1 \Big\rangle \end{aligned} \tag{31}$$

We cross them so

$$\left\| \vec{r}_u \times \vec{r}_v \right\| = \sqrt{\sin^2 v + \cos^2 v} = 1 \tag{32}$$

and our surface integral (29) becomes

$$\int_0^1 \int_0^{\pi/2} \sin u \, du \, dv = 1 \tag{33}$$

Flux

We return to where we first considered flux integrals where we introduced

$$\iint_S \vec{F} \cdot \vec{N} dS \tag{34}$$

Now, in terms of a parametric surface we have

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{\left\| \vec{r}_u \times \vec{r}_v \right\|}, \quad dS = \left\| \vec{r}_u \times \vec{r}_v \right\| dudv \tag{35}$$

so (34) becomes

$$\iint_{R_{uv}} \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dudv \tag{36}$$

Example 4

Find the flux over the unit sphere $x^2 + y^2 + z^2 = 1$ where $\vec{F}$ is given by $\vec{F} = \langle x, y, z \rangle$.

Soln.

We first parametrize the surface. Here we will use spherical polar coordinates so

$$\vec{r}(\theta,\phi) = \Big\langle \cos\theta\sin\phi,\, \sin\theta\sin\phi,\, \cos\phi \Big\rangle \qquad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi \tag{37}$$

Next we take derivatives so

$$\begin{aligned} \vec{r}_\theta &= \Big\langle -\sin\theta\sin\phi,\, \cos\theta\sin\phi,\, 0 \Big\rangle \\ \vec{r}_\phi &= \Big\langle \cos\theta\cos\phi,\, \sin\theta\cos\phi,\, -\sin\phi \Big\rangle \end{aligned} \tag{38}$$

Next we cross them so

$$\vec{r}_\theta \times \vec{r}_\phi = \Big\langle -\cos\theta\sin^2\phi,\; -\sin\theta\sin^2\phi, -\sin\theta\cos\phi \Big\rangle \tag{39}$$

As this normal points outward we multiply by -1.

$$\vec{r}_\theta \times \vec{r}_\phi = \Big\langle \cos\theta\sin^2\phi,\; \sin\theta\sin^2\phi, \sin\theta\cos\phi \Big\rangle \tag{40}$$

Next

$$\begin{aligned} \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) &= \Big\langle \cos\theta\sin\phi,\, \sin\theta\sin\phi,\, \cos\phi \Big\rangle \\ &\quad \cdot \Big\langle \cos\theta\sin^2\phi,\; \sin\theta\sin^2\phi, \sin\theta\cos\phi \Big\rangle \\ &= \sin\phi \end{aligned} \tag{41}$$

and the flux is given by

$$\int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta = 4\pi \tag{42}$$