Math 1497 - Calc 2

Trig integrals

$$\int \sin^m x \cos^n x \, dx \qquad \int \tan^m x \sec^n x \, dx$$

Ex $$\int_{x=0}^{\pi/4} \sin^4 x \cos^3 x \, dx = \int_{x=0}^{\pi/4} \sin^4 x \cos^2 x \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$x = 0 \quad u = \sin 0 = 0$

$x = \pi/4 \quad u = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\sin^4 x = u^4$

$\cos^2 x = 1 - \sin^2 x$

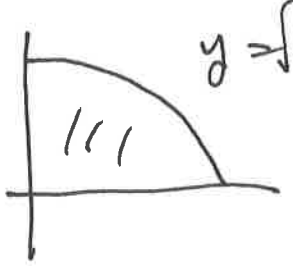
$= 1 - u^2$

So $$\int_0^{\frac{\sqrt{2}}{2}} u^4 (1-u^2) \, du$$

$$= \int_0^{\sqrt{2}/2} u^4 - u^6 \, du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} \Big|_0^{\sqrt{2}/2} = \frac{1}{5}\left(\frac{\sqrt{2}}{2}\right)^5 - \frac{1}{7}\left(\frac{\sqrt{2}}{2}\right)^7$$

Earlier we saw



$$A = \int_0^1 \sqrt{1-x^2}\,dx = \pi/4$$

so how do we integrate

$$\int \sqrt{1-x^2}\,dx$$

Similar Type's

$$\int \sqrt{1+x^2}\,dx \quad \& \quad \int \sqrt{x^2-1}\,dx$$

or, more general

$$\int \sqrt{a^2-x^2}\,dx\,, \quad \int \sqrt{x^2+a^2}\,dx, \quad \int \sqrt{x^2-a^2}\,dx$$

and variations of these.

For these types of integrals, we will use what is called a "trig Substitution". We will use the identities

$$\sin^2\theta + \cos^2\theta = 1, \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\int \sqrt{a^2 - x^2}\, dx \qquad \text{let } x = a\sin\theta \qquad dx = a\cos\theta\, d\theta$$

Now $a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta)$

$$= a^2\cos^2\theta$$

so $\sqrt{a^2 - x^3} = a\cos\theta$ square root gone!

$$\int \sqrt{x^2 + a^2}\, dx \qquad \text{let } x = a\tan\theta \qquad dx = a\sec^2\theta\, d\theta$$

so $x^2 + a^2 = a^2\tan^2\theta + a^2 = a^2(\tan^2\theta + 1)$

$$= a^2\sec^2\theta$$

so $\sqrt{x^2 + a^2} = a\sec\theta$ square root gone!

$\int \sqrt{x^2-a^2}\,dx$ let $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\,d\theta$

so $x^2-a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1)$

$= a^2\tan^2\theta$

so $\sqrt{x^2-a^2} = \sqrt{a^2\tan^2\theta} = a\tan\theta$ root is gone!

Ex 1 $\int \frac{x}{\sqrt{1-x^2}}\,dx$

1st let $u = 1-x^2$ so $du = -2x\,dx$

$$-\frac{1}{2}\int \frac{du}{u^{1/2}} = -u^{1/2} + C = -\sqrt{1-x^2} + C$$

Now let us use a trig sub. Here

$x = \sin\theta$ & $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta}$

$dx = \cos\theta\,d\theta$ $\quad -\sqrt{\cos^2\theta} = \cos\theta$

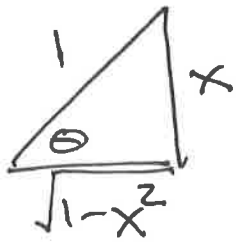
so integral becomes

$$\int \frac{\sin\theta}{\cos\theta}\cos\theta\,d\theta = \int \sin\theta\,d\theta$$

$$= -\cos\theta + C$$

Now we go back to x's. The sub. is

$x = \sin\theta$



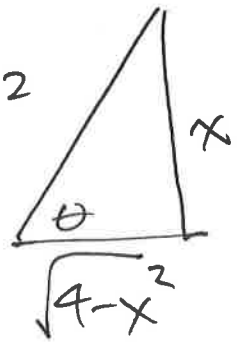
from triangle $\cos\theta = \frac{\sqrt{1-x^2}}{1}$

so $= -\cos\theta + C = -\sqrt{1-x^2} + C$ as before.

Ex 2 $\int \frac{dx}{x^2\sqrt{4-x^2}}$ $\quad x = 2\sin\theta$ $\quad dx = 2\cos\theta\,d\theta$ $\quad \sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$

$$\int \frac{2\cos\theta}{4\sin^2\theta \cdot 2\cos\theta}\,d\theta = \frac{1}{4}\int \csc^2\theta\,d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$



$\sin\theta = \frac{x}{2}$

Note: Part of triangle appears in integral — good sign!

Now $\cot\theta = \frac{\sqrt{4-x^2}}{x}$

$= -\frac{\sqrt{4-x^2}}{4x} + C$

Ex 3 $\int \frac{dx}{\sqrt{x^2+9}}$

$x = 3\tan\theta$

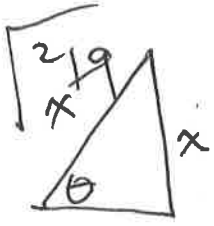
$dx = 3\sec^2\theta\, d\theta$

$\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9}$

$= 3\sec\theta$

$\int \frac{3\sec^2\theta\, d\theta}{3\sec\theta}$

$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$



$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{3}{\sqrt{x^2+9}}} = \frac{\sqrt{x^2+9}}{3}$

$= \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$

Ex 4 $\int \frac{\sqrt{x^2-16}}{x}\,dx$

let $x = 4\sec\theta$, $dx = 4\sec\theta\tan\theta\,d\theta$

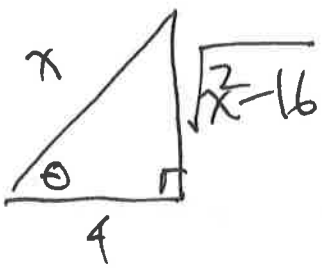
$x^2-16 = (4\sec\theta)^2 - 16 = 16(\sec^2\theta - 1)$

$= 16\tan^2\theta$

so $\int \frac{4\tan\theta}{4\sec\theta}\, 4\sec\theta\tan\theta\,d\theta$

$4\int \tan^2\theta\,d\theta = 4\int \sec^2\theta - 1\,d\theta$

$= 4(\tan\theta - \theta) + c$



$= 4\left(\frac{\sqrt{x^2-16}}{4} - \sec^{-1}\frac{x}{4}\right) + c$

what about θ ↑

$\sec\theta = \frac{x}{4}$ $\quad \sec\theta = \frac{x}{4}$ so $\theta = \sec^{-1}\frac{x}{4}$

$\cos\theta = \frac{4}{x}$