A Simple and Accurate Approximation to the SEP of Rectangular QAM in Arbitrary Nakagami-m Fading Channels

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***Abstract*— Recently, Karagiannidis presented a closed-form solution for an integral which can be used to compute the average symbol error probability of general order rectangular quadrature amplitude modulation (QAM) in Nakagami-m fading channels with integer fading parameters. In this letter, using an accurate exponential bound for the Gaussian Q-function, we derive a simple approximate solution for that integral. In particular, the solution can be used to compute the average SEP of general order rectangular QAM over arbitrary Nakagami-m fading. Numerical results are presented to verify the accuracy of the solution.**

***Index Terms*— Quadrature amplitude modulation (QAM), symbol error probability, Nakagami-m fading channels, bounds, Gaussian Q-function.**

I. Introduction

MICROWAVE and mobile high speed communication systems can be efficiently implemented by employing general order rectangular quadrature amplitude modulation (QAM) [1]. It is also used in asymmetric subscriber loop and telephone-line modems. The rectangular QAM constellations can be easily generated by using two independent pulse amplitude modulation (PAM) signals [2], [3].

Many publications have studied the error performance of rectangular QAM constellations [2]–[5]. Beaulieu in [2] presented a formula for the symbol error probability (SEP) of general order rectangular QAM, in additive white Gaussian noise (AWGN). In the same work, he derived a closed-form solution for the average of the product of two Gaussian Q-functions over slow Rayleigh fading. In [4] Yoon and Cho derived a closed-form expression for the exact bit error rate for Gray code mapped rectangular QAM. The M-ary square QAM, a popular modulation format used in many wireless applications, is a special case of rectangular QAM. Therefore the exact SEP or BER expressions of rectangular QAM provide more generalised performance evaluations for QAM modulation [4].

Recently, several authors have analyzed the performance of rectangular QAM constellations in Nakagami-m fading channels [3]–[5]. In [5] Liu and Hanzo derived the exact bit error rate of rectangular QAM by considering the co-channel interference effects and Nakagami-m fading. In another work, Karagiannidis investigated the average SEP of general order rectangular QAM [3]. In doing so, he calculated a closed-form expression for the average of the product of two Gaussian Q-functions over Nakagami-m fading. However, the solution which he presents applies only for integer m cases of Nakagami fading.

In this letter, we use a simple exponential bound introduced by Chiani *et al.* in [6] to evaluate the average of the product of two Gaussian Q-functions over Nakagami-m fading. Results indicate that this approximation is a convenient tool for calculating the SEP of rectangular QAM. The approximate and exact results agree closely. The approximate solution is quite simple and does not assume an integer m value for Nakagami fading. Therefore, it can be efficiently used to investigate the SEP of rectangular QAM constellations under different fading severity conditions. Furthermore, in many cases of practical interest, it is sufficient to have approximate solutions to facilitate mathematical manipulations [6], [7]. This is also true in many emerging technologies such as ad-hoc and sensor networks, where complex calculations must be avoided as much as possible especially at the node level [8].

II. Approximate Average of the Product of Two Q-Functions over Nakagami-m Fading

Consider evaluation of the following integral,

$$\Upsilon(a,b) = \int_0^\infty f_R(r)Q(ar)Q(br)dr \tag{1}$$

The Q-function is given by $1/\sqrt{2\pi}\int_x^\infty \exp(-y^2/2)dy$ and $f_R(r)$ is p.d.f of the Nakagami-m fading envelope given by [9]

$$f_R(r) = \frac{2m^m}{\Omega^m\Gamma(m)}r^{2m-1}\exp\left(-\frac{m}{\Omega}r^2\right) \tag{2}$$

where $\Omega = \mathcal{E}\{R^2\}$, $m = \Omega^2/\mathcal{E}\{(R^2-\Omega^2)^2\}$ and $\mathcal{E}\{\cdot\}$ denotes the statistical expectation operator. $\Gamma(x)$ is the Gamma function defined as $\int_0^\infty t^{z-1}\exp(-t)dt$ [10]. In the above p.d.f, m reflects different fading severity conditions. For example, $m=1$ represents the well known Rayleigh fading.

The integral involving the product of two Gaussian Q-functions in (1) can be considered as a generalized integral involved in many wireless communication performance analyses. For example, the SEP analysis of square M-QAM constellations require the computation of (1) with $a=b$.

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A. Approximate Evaluation of $\Upsilon(a, b)$

In [3, eq. 4] Karagiannidis derived the exact solution to (1) valid for integer m. The derivation follows the concept of [2, Appendix]. After evaluating the Nakagami-m cumulative density function, he has elegantly avoided the complexity issue by replacing the incomplete Gamma function with its finite series representation valid for integer values of m. In this letter we use a different approach to obtain an approximate solution to $\Upsilon(a, b)$, in terms of elementary functions. Our approach uses accurate exponential bounds for the Gaussian Q-function reported by Chiani *et al.* [6].

We note that Q-function is related to the complementary error function erfc(x) as

$$Q(x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{3}$$

Therefore, $\Upsilon(a, b)$ can be rewritten as

$$\Upsilon(a, b) = \frac{1}{4}\int_0^\infty \text{erfc}\left(\frac{ar}{\sqrt{2}}\right)\text{erfc}\left(\frac{br}{\sqrt{2}}\right) f_R(r)dr \tag{4}$$

A simple tight exponential bound for the erfc(x) function was presented by in [6, eq. 14] as

$$\text{erfc}(x) \simeq \frac{1}{6}e^{-x^2} + \frac{1}{2}e^{-4x^2/3} \tag{5}$$

The accuracy of the above tight upper bound was discussed in [6] where the authors mentioned its good agreement with erfc(x) for $x > 0.5$. Therefore, by using (5) we can approximate $\Upsilon(a, b)$ as

$$\begin{aligned}\Upsilon(a, b) \simeq \frac{m^m}{2\Omega^m\Gamma(m)}\Bigg(&\int_0^\infty \frac{1}{36}\exp\left(-\alpha_1 r^2\right)r^{2m-1}dr \\ &+ \int_0^\infty \frac{1}{4}\exp\left(-\alpha_2 r^2\right)r^{2m-1}dr \\ &+ \int_0^\infty \frac{1}{12}\exp\left(-\alpha_3 r^2\right)r^{2m-1}dr \\ &+ \int_0^\infty \frac{1}{12}\exp\left(-\alpha_4 r^2\right)r^{2m-1}dr\Bigg)\end{aligned} \tag{6}$$

where $\alpha_1 = (\Omega a^2 + \Omega b^2 + 2m)/2\Omega$, $\alpha_2 = (2\Omega a^2 + 2\Omega b^2 + 3m)/3\Omega$, $\alpha_3 = (3\Omega a^2 + 4\Omega b^2 + 6m)/6\Omega$ and $\alpha_4 = (4\Omega a^2 + 3\Omega b^2 + 6m)/6\Omega$. The four integrals involved in (6) can be simplified using the integral results of [10, eq. 3.478-1] to give

$$\Upsilon(a, b) \approx \frac{m^m}{16\Omega^m}\left(\frac{1}{9\alpha_1^m} + \frac{1}{\alpha_2^m} + \frac{1}{3\alpha_3^m} + \frac{1}{3\alpha_4^m}\right) \tag{7}$$

III. SEP of General Order Rectangular QAM in Nakagami-m Fading

In this section we use the approximate expression (7) for the $\Upsilon(a, b)$ to evaluate the average SEP of general order rectangular QAM in Nakagmi-m fading. The average SEP of general order rectangular QAM in Nakagami-m fading channels is given by [3]

$$P_e = \int_0^\infty P_e(rA_I, rA_Q)f_R(r)dr \tag{8}$$

and

$$\begin{aligned}P_e(A_I, A_Q) =& 2a_1\, Q(A_I) + 2a_2\, Q(A_Q) \\ &- 4a_1 a_2 Q(A_I)Q(A_Q)\end{aligned} \tag{9}$$

where $a_1 = \left(1 - \frac{1}{M_I}\right)$ and $a_2 = \left(1 - \frac{1}{M_Q}\right)$. M_I-PAM and M_Q-PAM are the in-phase and quadrature PAM signals.

Therefore, to calculate P_e, two integrals must be evaluated. Consider the following integral

$$\Psi(a) = \int_0^\infty Q(ar)f_R(r) \tag{10}$$

For this integral, a closed-form solution valid for integer values of m was reported in [11, eq. 5.17] as

$$\Psi(a) = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{a^2\Omega}{2m + a^2\Omega}}\sum_{k=0}^{m-1}\frac{(2k)!\, m^k}{(k!)^2(4m + 2a^2\Omega)^k} \tag{11}$$

Also using the integral result of [12, eq. 2.8.5-7] $\Psi(a)$ for non-integer values of m can be simplified to

$$\begin{aligned}\Psi(a) =& \frac{(2m)^{m-1}}{\sqrt{\pi}\Omega^m a^{2m}}\frac{\Gamma(m + \frac{1}{2})}{\Gamma(m)} \\ &\times\ {}_2F_1\left(m, m + \frac{1}{2}; m + 1; -\frac{2m}{\Omega a^2}\right)\end{aligned} \tag{12}$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is the Gauss hypergeometric function. Furthermore, if the approximate erfc(x) representation of (5) is employed $\Psi(a)$ is given by

$$\begin{aligned}\Psi(a) \simeq \frac{m^m}{\Omega^m\Gamma(m)}\Bigg(&\int_0^\infty \frac{1}{6}\exp\left(-\alpha_5 r^2\right)r^{2m-1}dr \\ &\int_0^\infty \frac{1}{2}\exp\left(-\alpha_6 r^2\right)r^{2m-1}dr\Bigg)\end{aligned} \tag{13}$$

where $\alpha_5 = (\Omega a^2 + 2m)/2\Omega$ and $\alpha_6 = (2\Omega a^2 + 3m)/3\Omega$. Eq. (13) can be simplified to

$$\Psi(a) \approx \frac{m^m}{4\Omega^m}\left(\frac{1}{3\alpha_5^m} + \frac{1}{\alpha_6^m}\right) \tag{14}$$

IV. Numerical Results

In this section we present several numerical results to verify the accuracy of the approximate analysis. All computations were performed using MATLAB. In Fig. 1 we compare the *exact* closed-form solution of [3, eq. 4] and the approximate solution of (7). In the case of non-integer m, the exact result was obtained by numerical integration. The same parameters considered in [3] for a $(8, 4)$ QAM constellation are used. For $(8, 4)$ QAM, the signal-to-noise ratio (SNR) is defined as

$$\frac{E_T}{\sigma_n^2} = \frac{(21 + 5\gamma^2)\Omega A_I^2}{2} \tag{15}$$

The approximate and exact $\Upsilon(a, b)$ values match remarkably for $m = 1$ and 3. However, for $m = 2.75$, the approximate approach slightly overestimates the result.

Fig. 2 shows the average SEP of rectangular QAM versus the average symbol energy-to-noise power E_T/σ_n^2 due to the approximate approach. In the first example a $(8, 4)$ QAM constellation is considered [2], [3]. “Approx. 1” used Eqs. (7)

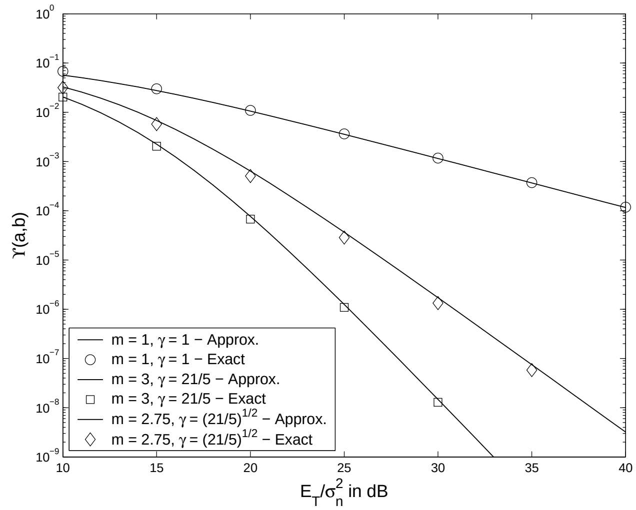

Fig. 1. Υ versus E_T/σ_n^2 for $(8,4)$ QAM constellation parameters.

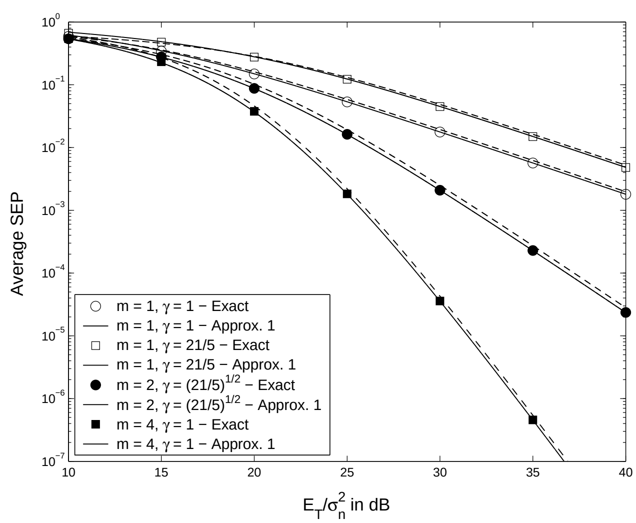

Fig. 2. Average SEP for $(8,4)$ QAM in Nakagami-m Fading. The dashed lines show results from (7) and (14).

and (11). In all cases considered, the average SEP predictions made using the approximate approach match very well with the exact closed-form results.

In the second example shown in Fig. 3, we consider a $(16,8)$ QAM constellation and investigate its average SEP over non-integer m Nakagami fading. For $(16,8)$ QAM, the SNR is defined as [2]

$$\frac{E_T}{\sigma_n^2} = \frac{(85 + 21\gamma^2)\Omega A_I^2}{2} \tag{16}$$

The average SEP computations were performed using (7) and (12) ("Approx. 1") and (7) and (14) ("Approx. 2"). The *exact* P_e was obtained from numerical integration. A decrease in fading severity indicated by a large m, significantly improves the performance depending on the SNR region and γ.

V. Conclusions

We have presented a simple approximate solution for the average of the product of two Gaussian Q-functions over

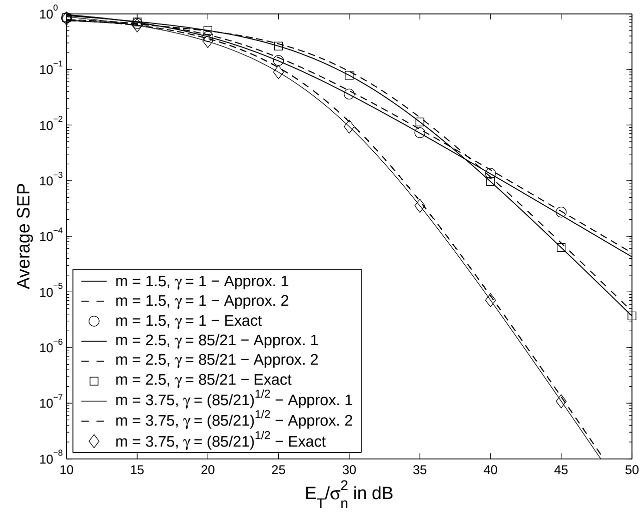

Fig. 3. Average SEP for $(16,8)$ QAM in Nakagami-m Fading.

Nakagami-m fading. This approximate solution is valid for both integer and non-integer values of m. We also investigated the average symbol error probability (SEP) of general rectangular QAM constellations over Nakagami-m fading channels. Numerical results confirm the accuracy of the average SEP predictions.

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