Math 6345 – Adv. ODE's

The last few classes we have been considering the stability of the zero solution of linear systems

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bar{x}$$

and have characterize the phase portraits depending on the eigen values.

Now we consider nonlinear systems

$$\frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y) \qquad — (1)$$

Critical Point (CP)

A critical pt of the system (1) is a constant sol^n $x = x_0$, $y = y_0$ such that

$$f(x_0, y_0) = 0, \quad g(x_0, y_0) = 0$$

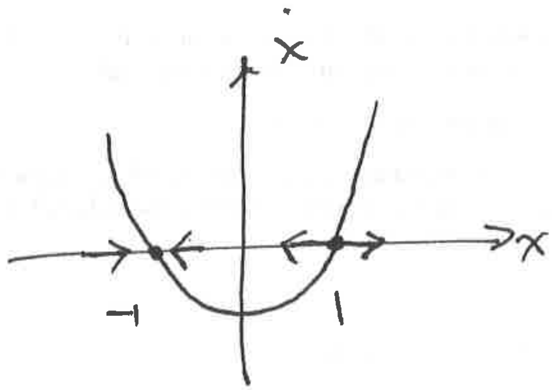
we ask – what is the stability of this (these) critical pts

As a motivation consider the single ODE

$$\frac{dx}{dt} = x^2 - 1$$

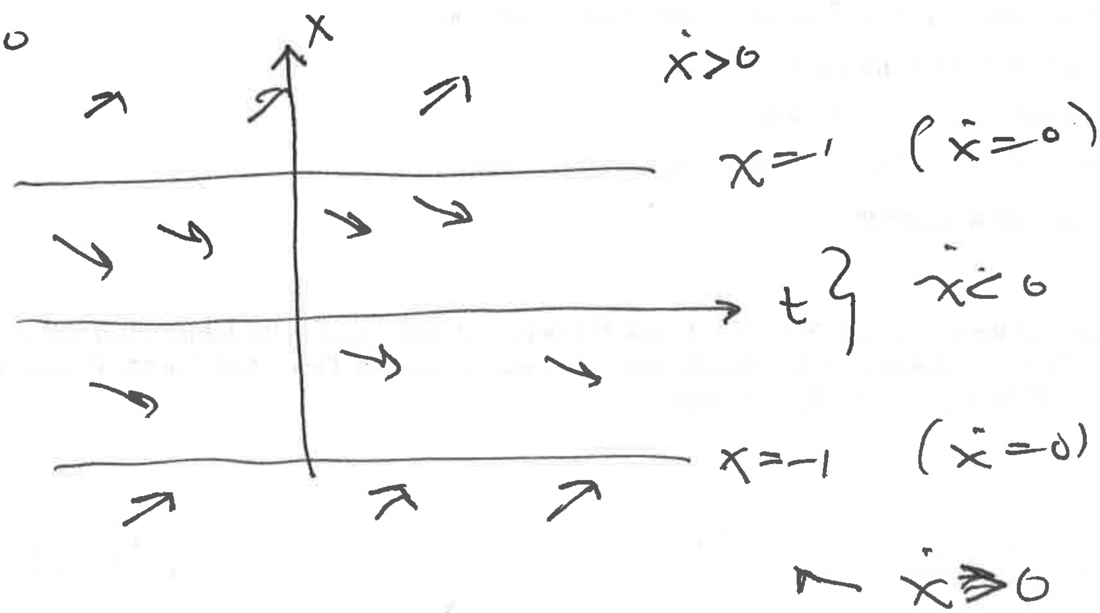
There are 2 critical pt $x = -1, 1$

If we draw the phase portrait (x v $\dot{x}$)



we see that $x=-1$ is asymptotically stable whereas $x=1$ is unstable

Also



so can we do this analytically (some-how)

Consider the 2 critical pt

$x = -1$ let $x = -1 + u$

so $\dot{x} = \dot{u}$

$$\dot{x} = x^2 - 1 \Rightarrow \quad \dot{u} = (-1+u)^2 - 1$$
$$= 1 - 2u + u^2 - 1$$
$$\dot{u} = -2u + u^2$$

Near $u = 0$ u^2 is small compared to $-2u$

so the eq^n $\dot{u} = -2u$

approximates the ~~second~~ sol^n near the critical pt so $u = c e^{-2t}$

and $x \simeq -1 + c e^{-2t}$

and we see that $x = -1$ is asym stable

Further if $x = 1 + u$ then $\dot{x} = x^2 - 1$

becomes $\dot{u} = 2u + u^2$

and similar near $u = 0$ $\quad \dot{u} \approx 2u$

So $\quad u = c e^{2t}$

and $\quad x = 1 + C e^{2t}$

and we see $\dot{u} = 2u$ is unstable so x is unstable

let's consider the actual sol^n

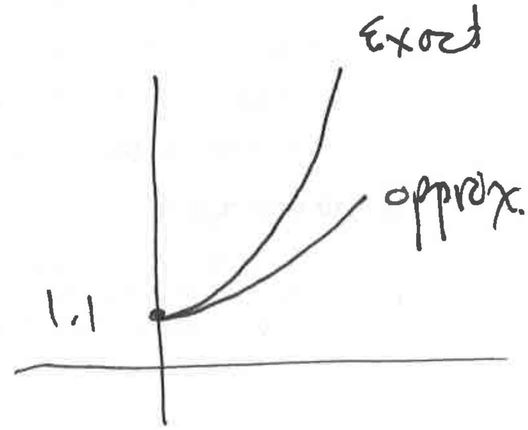
$\dot{x} = x^2 - 1 \quad$ sol^n $\quad x = \dfrac{c e^{-2t} - 1}{c e^{-2t} + 1}$

let's consider the initial pt $\quad x(0) = 1.1$

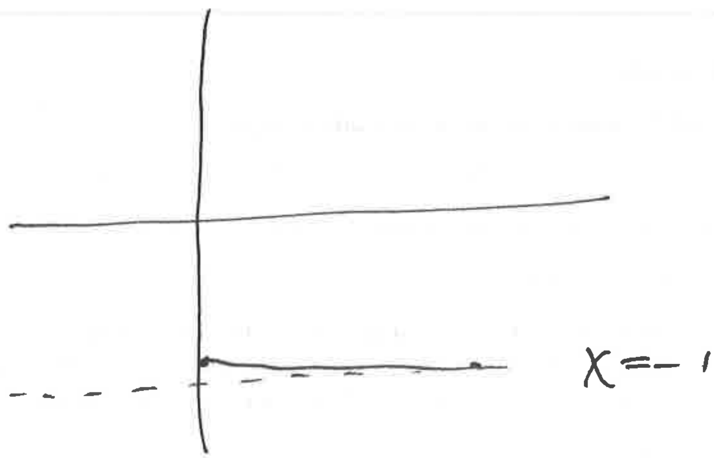
then $\quad 1.1 = \dfrac{c - 1}{c + 1} \Rightarrow c = -21$

So $\quad x = \dfrac{-21 e^{-2t} - 1}{-21 e^{-2t} + 1} \quad$ Exact

$x = +1 + .1 e^{2t} \quad$ App



~~Also~~ Also if $x(0) = -0.9$



$$x_e = \frac{.05263\, e^{-2t} - 1}{.05263\, e^{-2t} + 1} \qquad x_a = -1 + 0.1\, e^{-2t}$$

both are practical indistinguishable

Two Dimensional System

we now consider the 2D system

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y)$$

and some critical pt (x_0, y_0)

we linearize the system about this pt

$x = x_0 + u, \quad y = y_0 + v$

~~so~~ $\dot{x} = \dot{u}, \quad \dot{y} = \dot{v}$

so $\dot{x} = f(x,y) \Rightarrow \dot{u} = f(x_0+u, y_0+v)$

$$= f(x_0, y_0) + f_x(x_0, y_0)u + f_y(x_0, y_0)v + O(u^2, uv, v^2)$$

$$\dot{y} = g(x_0, y_0) + g_x(x_0, y_0)u + g_y(x_0, y_0)v + \text{hots}$$

$f(x_0, y_0) = g(x_0, y_0)$ so we consider the linear system

$$\begin{pmatrix} u \\ v \end{pmatrix}^{\cdot} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where the matrix is evaluated at the CP.

For short we will denote the linear system

as $\dot{x} = (D_x f)x$

where $D_x f = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$

Ex $\dot{x} = 2x - x^2 - xy \quad \dot{y} = xy - y$

cp $2x - x^2 - xy = 0 \qquad (x-1)y = 0$

the latter gives 2 case $x = 1,\ y = 0$

if $x = 1 \quad 2 - 1 - y = 0 \Rightarrow y = 1$ cp $(1,1)$

if $y = 0 \quad 2x - x^2 = 0 \Rightarrow x = 0, 2$ cp $(0,0), (2,0)$

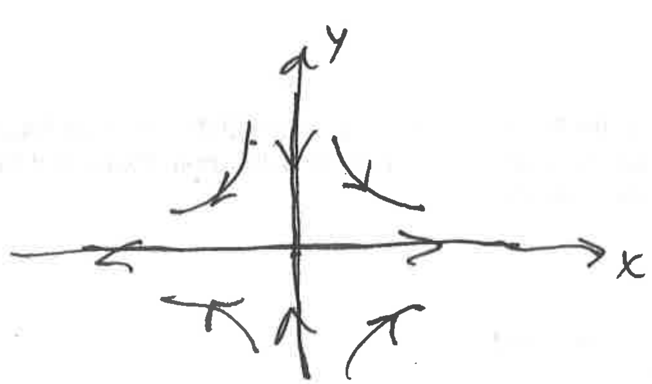
<u>Linearized System</u>

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \begin{pmatrix} 2-2x-y & -x \\ y & x-1 \end{pmatrix}\Bigg|_{cp}$$

at $(0,0)$ $\quad \dot{\bar{x}} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \bar{x}$ Eigenvalues $\lambda = 2, -1$

Saddle

Sol^n $\quad \bar{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$



cp (1, 1)

$$\dot{x} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \bar{x}$$

$\det(\lambda I - A) = 0$ $\quad \begin{vmatrix} \lambda+1 & 1 \\ -1 & \lambda \end{vmatrix} = 0 \quad \lambda^2 + \lambda + 1 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{3}\,i}{2}$$

decaying spiral

cp (3, 0)

$$\dot{x} = \begin{pmatrix} -3 & -2 \\ 0 & 1 \end{pmatrix} \bar{x}$$

$\det(\lambda I - A) = 0$ $\quad \begin{vmatrix} \lambda+3 & 2 \\ 0 & \lambda-1 \end{vmatrix} = 0 \quad (\lambda+3)(\lambda-1) = 0$

$\lambda = -3, 1$ saddle

sol^n $\quad \bar{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{t}$