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Research Article

Characterization of Fuzzy Soft Sets in Topological Spaces

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Abstract
In the present paper, we give a detailed study of soft fuzzy sets in topological spaces. We study some aspects like separability, connectedness, their generalizations and relationships between them.

Keywords: Fuzzy set; Characterization; Generalized fuzzy soft connected sets; Topological space.

Introduction

The notion of connectedness in fuzzy topological spaces has been studied by [1-3]. In fuzzy soft setting, connectedness has been introduced in [4-7]. In [8] they introduced the generalized fuzzy soft connectedness and generalized fuzzy soft Ci-connectedness (i=1, 2, 3, 4) in generalized fuzzy soft topological space and studied some of its basic properties. In this paper, we extend the notion of connectedness of fuzzy soft topological spaces to generalized fuzzy soft topological spaces in [9-11]. We introduce different notions of generalized fuzzy soft separated sets and study the relationship between them. The study is also devoted to introduce the different notions of connectedness in generalized fuzzy soft topological spaces and study the implications that exist between them [12,13]. Also, we study some characterizations of connectedness in generalized fuzzy soft setting [14-17].

Materials and methods

In this section, we give some basic concepts on generalized fuzzy soft sets, generalized fuzzy soft topology and generalized fuzzy soft continuous mappings which will be needed in the sequel.

Definition 2.1

Let X be a non-empty set. A fuzzy set A in X is defined by a membership function μA:$X \to [0,1]$ whose value $\mu A(x)$ represents the "grade of membership" of x in A for $x \in X$. The set of all fuzzy sets in a set X is denoted by IX, where I is the closed unit interval [0,1].

Definition 2.2

If A, $B \in IX$, then, we have:
(i) $A \leq B \Leftrightarrow (x) \leq \mu B(x), \forall\ x \in X$;
(ii) $A = B \Leftrightarrow (x) = \mu B(x), \forall\ x \in X$;
(iii) $C=A \vee B \Leftrightarrow (x)=\max(\mu A(x), \mu B(x)), \forall\ x \in X$;
(iv) $D=A \wedge B \Leftrightarrow (x)=\min(\mu A(x), \mu B(x)), \forall\ x \in X$;
(v) $E = AC \Leftrightarrow (x)=1- \mu A(x), \forall\ x \in X$.

Definition 2.3

Let X be an initial universe set and E be a set of parameters. Let (X) denotes the power set of X and $A \subseteq E$. A pair $(f, .)$ is called a soft set over X if f is a mapping from A into $P(X)$, i.e., f: $A \longrightarrow P(X)$. In other words, a soft set is a parameterized family of subsets of the set X. For $e \in A$, (e) may be considered as the set of e-approximate elements of the soft set $(f,)$.

Definition 2.4

Let X be an initial universe set and E be a set of parameters. Let $A \subseteq E$. A fuzzy soft set fA over X is a mapping from E to IX, i.e., fA:$E \longrightarrow IX$, where $(e) \neq \bar{0}$ if $e \in A \subseteq E$, and $fA(e)=\bar{0}$ if $e \notin A$, where $\bar{0}$ denotes the empty fuzzy set in X.

Definition 2.5

Let X be a universal set of elements and E be a universal set of parameters for X. Let F: $E \longrightarrow IX$ and μ be a fuzzy subset of E, i.e., μ: $E \longrightarrow X$. Let $F\mu$ be the mapping $F\mu$: $E \longrightarrow IX \times I$ defined as follows: (e) = $(F(e), \mu(e))$, where $F(e) \in IX$ and $\mu(e) \in I$. Then $F\mu$ is called a generalized fuzzy soft set ($GFSS$ in short) over $(X,)$. The

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family of all generalized fuzzy soft sets over $(X,)$ is denoted by $GFS(X,E)$.

Definition 2.6

Let $F\mu$ and $G\delta$ be two $GFSSs$ over $(X,)$. $F\mu$ is said to be a GFS subset of $G\delta$ or $G\delta$ is said to be a GFS super set of $F\mu$ denoted by $F\mu\subseteq G\delta$ if,
(i) μ is a fuzzy subset of δ;
(ii) (e) is also a fuzzy subset of (e),$\forall e \in E$.

Definition 2.7

Let $F\mu$ be a $GFSS$ over $(X,)$. The generalized fuzzy soft complement of $F\mu$, denoted by $F\mu c$, is defined by $F\mu c$=$G\delta$, where (e) =$\mu(e)$ and $G(e)$= $Fc(e)$, $\forall e\in E$.
Obviously (c)c=$F\mu$.

Definition 2.8

Let $F\mu$ and $G\delta$ be two $GFSSs$ over $(X,.)$. The generalized fuzzy soft union (GFS union, in short) of $F\mu$ and $G\delta$, denoted by $F\mu \sqcup G\delta$, is the $GFSS$ $H\nu$, defined as $H\nu$: $E \longrightarrow IX\times I$ such that $H\nu(e)$=$(H(e),\nu(e))$, where $H(e)$=$F(e)\vee G(e)$ and $\nu(e)$=$\mu(e)\vee\delta(e)$, $\forall e\in E$. Let {$(F\mu)$, $\lambda\in\nabla$}, where ∇ is an index set, be a family of $GFSSs$. The GFS union of these family, denoted by $\sqcup\lambda\in\Lambda(F\mu)\lambda$, is The $GFSS$ $H\nu$, defined as $H\nu$: $E \longrightarrow IX\times I$ such that $H\nu(e)$=$(H(e),\nu(e))$, where $H(e)$=$\vee\lambda\in\nabla(F(e))\lambda$, and $\nu(e)$=$\vee\lambda\in\nabla(\mu(e))\lambda$, $\forall e\in E$.

Definition 2.9

Let $F\mu$ and $G\delta$ be two $GFSSs$ over $(X,.)$. The generalized fuzzy soft Intersection (GFS Intersection, in short) of $F\mu$ and $G\delta$, denoted by $F\mu \sqcap G\delta$, is the $GFSS$ $M\sigma$, defined as $M\sigma$: $E \longrightarrow IX\times I$ such that $M\sigma(e)$=$(M(e),\sigma(e))$, where $M(e)$=$F(e)\wedge G(e)$ and $\sigma(e)$=$\mu(e)\wedge\delta(e)$, $\forall e\in E$. Let {$(F\mu)\lambda$,$\lambda\in\nabla$}, where ∇ is an index set, be a family of $GFSSs$. The GFS Intersection of these family, denoted by $\sqcap\lambda\in\nabla(F\mu)\lambda$, is the $GFSS$ $M\sigma$, defined as $M\sigma$: $E \longrightarrow IX\times I$ such that $M\sigma(e)$=$(M(e),\sigma(e))$, where $M(e)$=$\wedge\lambda\in\nabla(F(e))\lambda$, and $\sigma(e)$=$\wedge\lambda\in\nabla(\mu(e))\lambda$, $\forall e\in E$.

Theorem 2.10

Let {$(F\mu)\lambda$,$\lambda\in\nabla$}$\subseteq GFSS(X,E)$. Then the following statements [3] hold:
(i) $[\sqcup\lambda\in\nabla(F\mu)\lambda ,\lambda\in\nabla]c$=$\sqcap\lambda\in\nabla(F\mu)\lambda c$,
(ii) $[\sqcap\lambda\in\nabla(F\mu)\lambda ,\lambda\in\nabla]c$=$\sqcup\lambda\in\nabla(F\mu)\lambda c$.

Definition 2.11

A $GFSS$ is said to be a generalized null fuzzy soft set, denoted by $\tilde{0}\theta$, if $\tilde{0}\theta$:$E \longrightarrow IX\times I$ such that $\tilde{0}\theta(e)$=$(\tilde{0}(e),\theta(e))$ where $\tilde{0}(e)$=$\bar{0}$ $\forall e\in E$ and $\theta(e)$=0 $\forall e\in E$ (Where $\bar{0}(x)$=0,$\forall x\in X$).

Definition 2.13

A $GFSS$ is said to be a generalized absolute fuzzy soft set, denoted by $\tilde{1}\Delta$, if $\tilde{1}\Delta$: $E \longrightarrow IX\times I$, where $\tilde{1}\Delta(e)$=$(\tilde{1}(e),\Delta(e))$ is defined by $\tilde{1}(e)$=$\bar{1}$,$\forall$ $e\in E$ and $\Delta(e)$=1,$\forall e\in E$ (Where $\bar{1}(x)$ = 1,$\forall x\in X$).

Definition 2.14

Let T be a collection of generalized fuzzy soft sets over $(X,)$. Then T is said to be a generalized fuzzy soft topology (GFS topology in short) over $(X,)$ if the following conditions are satisfied:
(i) $\tilde{0}\theta$ and $\tilde{1}\Delta$ are in T;
(ii) Arbitrary GFS unions of members of T belong to T;
(iii) Finite GFS intersections of members of T belong to T.
The triple $(X,,)$ is called a generalized fuzzy soft topological space ($GFST$-space in short) over (X,E). The members of T are called generalized fuzzy soft open sets [S open in short] in (X,T,E).

Definition 2.15

Let $(X,,)$ be a $GFST$-space. A $GFSS$ $F\mu$ over $(X,)$ is said to be a generalized fuzzy soft closed set in X [GFS closed in short], if its complement $F\mu c$ is GFS open. The collection of all GFS closed sets will be denoted by Tc.

Definition 2.16

Let $(X,)$ be a $GFST$-space and $F\mu\in GFSS(X,E)$. The generalized fuzzy soft closure of $F\mu$, denoted by $(F\mu)$, is the intersection of all GFS closed supper sets of $F\mu$. i.e., $(F\mu)$=$\sqcap${$H\nu$: $H\nu$ $\in Tc$,$F\mu\sqsubseteq H\nu$}. Clearly, $(F\mu)$ is the smallest GFS closed set over (X,E) which contains $F\mu$.

Definition 2.17

The generalized fuzzy soft set $F\mu \in G(X,E)$ is called a generalized fuzzy soft point (GFS point in short) if there exist $e\in E$ and $x\in X$ such that:
(i) $(e)(x)$=α ($0<\alpha\leq 1$) and $F(e)(y)$=0 for all $y\in X-\{x\}$,

(ii) $(e)=\lambda$ $(0<\lambda\leq 1)$ and $\mu(e')=0$ for all $e'\in E-\{e\}$. We denote this generalized fuzzy soft point $F\mu=(x\alpha,e\lambda)$.
$(x,)$ and $(\alpha,)$ are called respectively, the support and the value of $(x\alpha,e\lambda)$.

Definition 2.18

Let $F\mu$ be a $GFSS$ over $(X,)$. We say that $(x\alpha,)\tilde{\in}F\mu$ read as $(x\alpha,e\lambda)$ belongs to the $GFSS$ $F\mu$ if for the element $e\in E$, $\alpha\leq F(e)(x)$ and $\lambda\leq\mu(e)$.

Definition 2.19

For any two $GFSSs$ $F\mu$ and $G\delta$ over $(X,)$. $F\mu$ is said to be a generalized fuzzy soft quasi-coincident with $G\delta$, denoted by $F\mu qG\delta$, if there exist $e\in E$ and $x\in X$ such that $(e)(x)+G(e)(x)>1$ and μ (e) + δ (e) > 1. If $F\mu$ is not generalized fuzzy soft quasi-coincident with $G\delta$, then we write $F\mu\bar{q}G\delta$, i.e., for every $e\in E$ and $x\in X$, $(e)(x)+G(e)(x)\leq 1$ or for every $e\in E$ and $x\in X$, μ $(e)+\delta$ $(e)\leq 1$.

Definition 2.20

Let $(x\alpha,)$ be a GFS point and $F\mu$ be a $GFSS$ over (X,E). $(x\alpha,)$ is said to be generalized fuzzy soft quasi-coincident with $F\mu$, denoted by $(x\alpha,e\lambda)qF\mu$, if and only if there exists an element $e\in E$ such that $\alpha+F(e)(x)>1$ and $\lambda+\mu$ $(e)>1$.

Theorem 2.21

Let $F\mu$ and $G\delta$ are $GFSSs$ over $(X,)$. Then the following hold[17]:
(i)$F\mu\sqsubseteq G\delta\Leftrightarrow F\mu\bar{q}(G\delta)c$;
(ii) $F\mu qG\delta\Rightarrow F\mu\sqcap G\delta\neq\tilde{0}\theta$;
(iii) $(x\alpha,)\bar{q}F\mu\Leftrightarrow(x\alpha,e\lambda)\tilde{\in}(F\mu)c$;
(iv) $F\mu\bar{q}(F\mu)c$.

Definition 2.22

Let $GF(X,E)$ and $GFSS(Y,K)$ be the families of all generalized fuzzy soft sets over (X,E) and (Y,K) , respectively. Let $u:X\longrightarrow Y$ and $p:E\longrightarrow K$ be two functions. Then a mapping fup : $GFSS(X,E)\longrightarrow GFSS(Y,K)$ is defined as follows: for a generalized fuzzy soft set $F\mu\in GFSS(X,E)$,$\forall$ $k\in p(E)\subseteq K$ and $y\in Y$, $fup(F\mu)(k)(y)=\{(\vee x\in u-1(y)\vee e\in p-1(k)$ $F(e)(x)$,$\vee e\in p-1(k)\mu(e))$ if $u-1(y)\neq\varphi,p-1(k)\neq\varphi$,(0,0), $otherwise$.
fup is called a generalized fuzzy soft mapping [GFS mapping in short] and $f(F\mu)$ is called a GFS image of a $GFSS$ $F\mu$.

Definition 2.23

Let u: $X\longrightarrow Y$ and p: $E\longrightarrow K$ be mappings. Let f : $GFSS(X,E)\longrightarrow GFSS(Y,K)$ be a GFS mapping and $G\delta\in GFSS(Y,K)$. Then, $fup^{-1}(G\delta)\in GFSS(X,E)$, defined as follows:
$fup^{-1}(G\delta)(e)(x)=(G(p(e))(u(x)),\delta(p(e)))$, for $e\in E$, $x\in X$.
$fup^{-1}(G\delta)$ is called a GFS inverse image of $G\delta$.
If u and p are injective then the generalized fuzzy soft mapping fup is said to be injective. If u and p are surjective then the generalized fuzzy soft mapping fup is said to be surjective. The generalized fuzzy soft mapping fup is called constant, if u and p are constant.

Definition 2.24

Let $(X,T1,E)$ and $(Y,T2,K)$ be two $GFST$-spaces, and fup : $(X,T1,E)\longrightarrow(Y,T2,K)$ be a GFS mapping. Then fup is called
(i) generalized fuzzy soft continuous [*GFS*-continuous in short] if $fup-1(G\delta)\in T_1$ for all $G\delta\in T_2$.
(ii) generalized fuzzy soft open [*GFS* open in short] if $fup(F\mu)\in T_2$ for each $F\mu\in T_1$.

Definition 2.25

Let $(X,,)$ be a $GFST$-space and $F\mu\in GFS(X,E)$. Then, $F\mu$ is called
i. $GFSC1$-connected if and only if it does not exist two nonvoid GFS open sets $H\nu$ and $K\gamma$ such that $F\mu\sqsubseteq H\nu\sqcup K\gamma$, $H\nu\sqcap K\gamma\sqsubseteq F\mu c$, $F\mu\sqcap H\nu\neq\tilde{0}\theta$ and $F\mu\sqcap K\gamma\neq\tilde{0}\theta$.
ii. $GFSC2$-connected if and only if it does not exist two nonvoid GFS open sets $H\nu$ and $K\gamma$ such that $F\mu\sqsubseteq H\nu\sqcup K\gamma$, $F\mu\sqcap H\nu\sqcap K\gamma=\tilde{0}\theta$, $F\mu\sqcap H\nu\neq\tilde{0}\theta$ and $F\mu\sqcap K\gamma\neq\tilde{0}\theta$.
iii. $GFSC3$-connected if and only if it does not exist two nonvoid GFS open sets $H\nu$ and $K\gamma$ such that $F\mu\sqsubseteq H\nu\sqcup K\gamma$, $H\nu\sqcap K\gamma\sqsubseteq F\mu c$, $H\nu\not\sqsubseteq F\mu c$ and $K\gamma\not\sqsubseteq F\mu c$.
iv. $GFSC4$-connected if and only if it does not exist two nonvoid GFS open sets $H\nu$ and $K\gamma$ such that $F\mu\sqsubseteq H\nu\sqcup K\gamma$, $F\mu\sqcap H\nu\sqcap K\gamma=\tilde{0}\theta$, $H\nu\not\sqsubseteq F\mu c$ and $K\gamma\not\sqsubseteq F\mu c$.
Otherwise, $F\mu$ is called not $GFSCi$-connected set for i=1, 2, 3, 4.

Remark 2.26

In the above definition, if we take $\tilde{1}\Delta$ instead of F, then the $GFST$-space $(X,,E)$ is called $GFSCi$-connected space (i=1,2,3,4).

Definition 2.27

Two non-null $GFSS$ sets $F\mu$ and $G\delta$ in $GFST$-space $(X,)$ are said to be generalized fuzzy soft $Q-$separated [GFS $Q-$separated, in short] if $cl(F\mu)\sqcap G\delta = F\mu \sqcap cl(G\delta) = \tilde{0}\theta$.

Definition 2.28

Two non- null $GFSSs$ $F\mu$ and $G\delta$ in $GFST$-space $(X,,)$ are said to be generalized fuzzy soft weakly separated [in short, GFS weakly separated] if $cl(F\mu)qG\delta$ and $F\mu q cl(G\delta)$.

Definition 2.29

Two non- null $GFSSs$ $F\mu$ and $G\delta$ in $GFST$-space $(X,,)$ are said to be generalized fuzzy soft separated [in short, GFS separated] if there exist GFS open sets $H\nu$ and $K\gamma$ such that $F\mu \sqsubseteq H\nu, G\delta \sqsubseteq K\gamma$ and $F\mu \sqcap K\gamma = G\delta \sqcap H\nu = \tilde{0}\theta$.

Definition 2.30

Let $F\mu \in G(X,E)$. The generalized fuzzy soft support (in short, GFS support) of $F\mu$ defined by $S(F\mu)$ is the set, $S(F\mu) = \{x \in X, e \in E : F(e)(x) > 0$ and $\mu(e) > 0\}$.

Definition 2.31

Two non- null $GFSSs$ $F\mu$ and $G\delta$ are said to be GFS quasi-coincident with respect to $F\mu$ if $F(e)(x)+G(e)(x)>1$ and $\mu(e)+\delta(e)>1$ for every $x,e \in S(F\mu)$.

Definition 2.32

Two non- null $GFSSs$ $F\mu$ and $G\delta$ in a $GFST$-space $(X,,)$ are said to be generalized fuzzy soft strongly separated [in short, GFS strongly separated] if there exist GFS open sets $H\nu$ and $K\gamma$ such that

i. $F\mu \sqsubseteq H\nu, \sqsubseteq K\gamma$ and $F\mu \sqcap K\gamma = G\delta \sqcap H\nu = \tilde{0}\theta$,

ii. $F\mu$ and $H\nu$ are GFS quasi-coincident with respect to $F\mu$,

iii. $G\delta$ and $K\gamma$ are GFS quasi-coincident with respect to $G\delta$.

Definition 2.33

Let $(X,.)$ be a $GFST-$space over (X,E) and $G\delta$ be GFS subset of (X,E). Then $TG\delta = \{G\delta \sqcap F\mu : F\mu \in T\ \}$ is called a GFS relative topology and $(G\delta, TG\delta, E)$ is called a GFS subspace of (X,T,E). If $G\delta \in T$ (resp, $G\delta \in Tc$) then $(G\delta,\delta,E)$ is called generalized fuzzy soft open (resp. closed) subspace of (X,T,E).

Definition 2.34

A $GFSS$ $F\mu$ in a $GFST$-space (X,T,E) is called GFS Q-connected set if there does not two non-null GFS Q-separated sets $H\nu$ and $K\gamma$ such that $F\mu = H\nu \sqcup K\gamma$, Otherwise, $F\mu$ is called not GFS Q-connected set.

Definition 2.35

A $GFSS$ $F\mu$ in a $GFST$-space (X,T,E) is called GFS weakly-connected set if there does not two non-null GFS weakly separated sets $H\nu$ and $K\gamma$ such that $F\mu = H\nu \sqcup K\gamma$, Otherwise, $F\mu$ is called not GFS weakly-connected set.

Definition 2.36

A $GFSS$ $F\mu$ in a $GFST$-space $(X,\ E)$ is called GFS $s-$connected (respectively, GFS strongly-connected) set if there does not two non-null GFS separated (respectively, not strongly separated) sets $H\nu$ and $K\gamma$ such that $F\mu = H\nu \sqcup K\gamma$, Otherwise, $F\mu$ is called not GFS s-connected (respectively, GFS strongly-connected) set.

Definition 2.37

A $GFSS$ $F\mu$ in a $GFST$-space $(X,\ E)$ is called generalized fuzzy soft clopen set (GFS clopen set, in shoft) if $F\mu, \mu c \in T$.

Definition 2.38

A $GFSS$ $F\mu$ in a $GFST$-space $(X,\ E)$ is called GFS clopen-connected set in $(X,)$ if there does not exist any non-null proper GFS clopen set in $(F\mu, TF\mu, E)$. In this definitions, if we take $\tilde{1}\Delta$ instead of $F\mu$, then the $GFST$-space $(X,,E)$ is called GFS Q-connected (respectively, GFS weakly-connected, GFS s-connected, GFS strongly-connected, GFS clopen-connected) space.

Results and discussions

At this juncture, we will introduce different notions of generalized fuzzy soft separated sets and study the relation between these notions. We also carry out characterizations of the generalized fuzzy soft separated sets.

Theorem 3.1

Let $(X,,)$ be a $GFST$-space, $F\mu$ and $G\delta$ be two GFS closed sets in (X,E). Then $F\mu$ and $G\delta$ are GFS $Q-$separated sets if and only if $F\mu \sqcap G\delta = \tilde{0}\theta$.

Proof. Suppose that $F\mu$ and $G\delta$ are GFS Q-separated sets. Then $(F\mu)\sqcap G\delta = F\mu \sqcap cl(G\delta) = \tilde{0}\theta$. Since $F\mu$ and $G\delta$ are GFS closed sets then, $F\mu \sqcap G\delta = \tilde{0}\theta$. Conversely, let $F\mu \sqcap G\delta = \tilde{0}\theta$. Since

$F\mu$ and $G\delta$ are GFS closed sets, then $(F\mu)\sqcap G\delta = F\mu \sqcap G\delta = \tilde{0}\theta$ and $F\mu \sqcap cl(G\delta) = F\mu \sqcap G\delta = \tilde{0}\theta$. It follows that, $F\mu$ and $G\delta$ are GFS Q-separated sets.

Theorem 3.2

Let $H\nu$, be GFS Q-separated sets of $GFST$-space (X,T,E) and $F\mu \sqsubseteq H\nu, G\delta \sqsubseteq K\gamma$.Then, $F\mu, G\delta$ are $GFSQ-$separated sets.

Proof. Let $F\mu \sqsubseteq H\nu$. Then, $(F\mu) \sqsubseteq cl(H\nu)$. It follows that, $(F\mu)\sqcap G\delta \sqsubseteq cl(F\mu)\sqcap K\gamma \sqsubseteq cl(H\nu)\sqcap K\gamma = \tilde{0}\theta$. Also, since $G\delta \sqsubseteq K\gamma$. Then, $(G\delta) \sqsubseteq cl(K\gamma)$. Hence, $F\mu \sqcap (G\delta) \sqsubseteq H\nu \sqcap cl(K\gamma) = \tilde{0}\theta$. Thus $F\mu$, are $GFSQ-$separated sets.

Theorem 3.3

Let $(X,,)$ be a $GFST$-space and $F\mu, G\delta \in GFS(X,E)$. Then, $F\mu$ and $G\delta$ are GFS weakly separated sets if and only if there exist GFS open sets $H\nu$ and $K\gamma$ such that $F\mu \sqsubseteq H\nu, \sqsubseteq K\gamma$, and $F\mu q K\gamma$ and $G\delta q H\nu$.

Proof. Let $F\mu$ and $G\delta$ are GFS weakly separated sets in $(X,,)$. Then $(F\mu) q G\delta$ and $F\mu q cl(G\delta)$. Therefore, $G\delta \sqsubseteq [cl(F\mu)]c$ and $F\mu \sqsubseteq [cl(G\delta)]c$. Taking $H\nu = [cl(G\delta)]c$ and $K\gamma = [cl(F\mu)]c$. Then, $H\nu, \in T$, $F\mu q K\gamma$ and $G\delta q H\nu$. The converse is obvious.

Theorem 3.4

Let $F\mu$ and $G\delta$ are GFS Q-separated (respectively, separated, strongly separated, weakly separated) sets in $(X,)$ and $H\nu \sqsubseteq F\mu, \gamma \sqsubseteq G\delta$. Then, $H\nu$ and $K\gamma$ are GFS Q-separated (respectively, separated, strongly separated, weakly separated) sets in $(X, .)$.

Proof. As a sample, we will prove the case GFS $Q-$separated. Let $F\mu$ and $G\delta$ are GFS $Q-$separated in $(X,)$. Then, $(F\mu)\sqcap G\delta = F\mu \sqcap cl(G\delta) = \tilde{0}\theta$. Since $H\nu \sqsubseteq F\mu, \sqsubseteq G\delta$, then $(H\nu)\sqcap K\gamma = H\nu \sqcap cl(K\gamma) = \tilde{0}\theta$, therefore, $H\nu$ and $G\delta$ are GFS $Q-$separated set in (X,E).

Theorem 3.5

Let $(X,,)$ be a $GFST$-space and $F\mu, G\delta \in GFS(X,E)$. Then, $F\mu$ and $G\delta$ are GFS $Q-$separated in $(X,)$ if and only if there exist GFS closed sets $H\nu$ and $K\gamma$ such that $F\mu \sqsubseteq H\nu, \delta \sqsubseteq K\gamma$ and $F\mu \sqcap K\gamma = G\delta \sqcap H\nu = \tilde{0}\theta$.

Proof. Let $F\mu$ and $G\delta$ are GFS $Q-$separated in $(X,)$. Then, $(F\mu)\sqcap G\delta = F\mu \sqcap cl(G\delta) = \tilde{0}\theta$. Taking $H\nu = (F\mu)$ and $K\gamma = cl(G\delta)$. Therefore, $H\nu$ and $K\gamma$ are GFS closed sets in $(X,)$ such that $F\mu \sqsubseteq H\nu, \delta \sqsubseteq K\gamma$ and $F\mu \sqcap K\gamma = G\delta \sqcap H\nu = \tilde{0}\theta$. The converse is obvious.

Theorem 3.6

Let $(X,,E)$ be a $GFST$-space and $G\delta \sqsubseteq F\mu \tilde{\in} GFSS(X,E)$.Then, $clF\mu(G\delta) = cl(G\delta)\sqcap F\mu$, where $clF(G\delta)$ denotes the GFS closure in the GFS subspace $(F\mu, TF\mu, E)$.

Proof. We know $(G\delta)$ is GFS closed set in $(X,T,E) \Longrightarrow cl(G\delta)\sqcap F\mu$ is GFS closed set in $(F\mu, TF\mu, E)$. Now, $G\delta \sqsubseteq cl(G\delta)\sqcap F\mu$ and GFS closure of $G\delta$ in $(F\mu, TF\mu, E)$ is the smallest GFS closed set containing $G\delta$, so, GFS closure of $G\delta$ in $(F\mu, TF\mu, E)$ is contained in $cl(G\delta)\sqcap F\mu$ i.e., $clF\mu(G\delta) \sqsubseteq cl(G\delta)\sqcap F\mu$.

Conversely, let $clF(G\delta)$ be a GFS closure of $G\delta$ in $(F\mu, TF\mu, E)$. Since, $cl(G\delta)$ is GFS closed set in $(F\mu, TF\mu, E) \Longrightarrow clF\mu(G\delta) = K\gamma \sqcap F\mu$ where $K\gamma$ is GFS closed set in (X,T,E). Then, $K\gamma$ is GFS closed set containing $G\delta \Longrightarrow (G\delta) \sqsubseteq K\gamma \Longrightarrow cl(G\delta)\sqcap F\mu \sqsubseteq K\gamma \sqcap F\mu \sqsubseteq clF\mu(G\delta)$.

Theorem 3.7

Let (X, E) be a $GFST$-space and $G\delta \sqsubseteq F\mu \in GF(X,E)$. If $H\nu$ and $K\gamma$ are GFS separated (respectively, $Q-$separated, strongly separated, weakly separated) in $(F\mu, TF\mu, E)$, then $H\nu$ and $K\gamma$ are GFS separated (respectively, Q-separated, strongly separated, weakly separated) in $(G\delta, TG\delta, E)$.

Proof. As a sample, we will prove the case GFS weakly separated. Let $H\nu$ and $K\gamma$ be GFS weakly separated sets in $(F\mu, \mu, E)$. Then, $cl(H\nu) q K\gamma$ and $H\nu q clF\mu(K\gamma)$. Since, $G\delta \sqsubseteq F\mu$.Then, $clG\delta(H\nu) = clF\mu(H\nu)\sqcap G\delta \sqsubseteq clF\mu(H\nu)$ and $clG\delta(K\gamma) = clF\mu(K\gamma)\sqcap G\delta \sqsubseteq clF\mu(K\gamma)$. Therefore, $cl(H\nu) q K\gamma$ and $H\nu q clG\delta(K\gamma)$. Thus, $H\nu$ and $K\gamma$ be GFS weakly separated in $(G\delta, \delta, E)$. At this point, we introduce different notions of connectedness of $GFSSs$ and study the relation between these notions. We also characterize generalized fuzzy soft connected sets.

Theorem 3.8

The GFS-weakly connected set in $(X,)$ is a GFS Q-connected.

Proof. Let $F\mu$ be a GFS-weakly connected set in $(X,)$. Suppose $F\mu$ is not a GFS Q-connected. Then, there exist two non-null GFS Q-separated sets $H\nu$ and $K\gamma$ such that $F\mu = H\nu \sqcup K\gamma$. Now we have $H\nu$ and $K\gamma$ are non-null GFS weakly separated sets in $(X,)$ such that $F\mu = H\nu \sqcup K\gamma$.

Therefore, $F\mu$ is not a GFS-weakly connected set in $(X,)$, a contradiction. Hence, $F\mu$ is a GFS Q-connected.

Remark 3.10.

A GFS Q-connected set may not be GFS weakly-connected

Theorem 3.11

A $GFSC1$-connected set in $(X,)$ is GFS weakly-connected.

Proof. Let $F\mu$ be a $GFSC1$-connected set in $(X,.)$. Suppose $F\mu$ is not GFS weakly-connected. Then, there exist two nonvoid GFS weakly separated sets $H\nu$ and $K\gamma$ such that $F\mu = H\nu \sqcup K\gamma$. By Theorem 3.3, there exist GFS open sets $M\psi$ and $N\eta$ such that $H\nu \sqsubseteq M\psi, \sqsubseteq N\eta$, $H\nu q N\eta$ and $M\psi q K\gamma$. Then, $F\mu \sqsubseteq M\psi \sqcup N\eta$. Also, $F\mu \sqcap M\psi \neq \tilde{0}\theta$. For, if $F\mu \sqcap M\psi = \tilde{0}\theta$, then $F\mu \sqcap H\nu = \tilde{0}\theta$ so that $H\nu = \tilde{0}\theta$ (since $F\mu = H\nu \sqcup K\gamma$ implies that $H\nu \sqsubseteq F\mu$), which contradiction that $H\nu$ is a non-null. Similarly, $F\mu \sqcap N\eta \neq \tilde{0}\theta$. Also, $M\psi \sqcap N\eta \sqsubseteq (F\mu)c$. For, if $M\psi \sqcap N\eta \not\sqsubseteq F\mu c$, then there exist $x \in X, \in E$ such that $M(e)(x) > 1 - F(e)(x)$, $\psi(e) > 1 - \mu(e)$ and $N(e)(x) > 1 - F(e)(x)$, $\eta(e) > 1 - \mu(e)$. This means $M(e)(x) + F(e)(x) > 1$, $\psi(e) + \mu(e) > 1$ and $N(e)(x) + F(e)(x) > 1$, $\eta(e) + \mu(e) > 1$. Since, $F\mu = H\nu \sqcup K\gamma$, then $M(e)(x) + H(e)(x) > 1$, $\psi(e) + \nu(e) > 1$ or $M(e)(x) + K(e)(x) > 1$, $\psi(e) + \gamma(e) > 1$ and
$N(e)(x) + H(e)(x) > 1$, $\eta(e) + \nu(e) > 1$ or $N(e)(x) + K(e)(x) > 1$, $\eta(e) + \gamma(e) > 1$. Hence, $(M\psi q H\nu$ or $M\psi q K\gamma)$ and $(N\eta q H\nu$ or $N\eta q K\gamma)$. This a contradiction. So, $F\mu$ is a GFS weakly-connected.

Remark 3.12

The GFS weakly-connected set may not be a $GFSC1$-connected.

Theorem 3.13

A GFS weakly-connected set in $(X,)$ is $GFSC2$-connected.

Proof. Let $F\mu$ be a GFS weakly-connected set in $(X,)$. Suppose $F\mu$ is not $GFSC2$-connected. Then, there exist $H\nu$ and $K\gamma \in T$ such that $F\mu \sqsubseteq H\nu \sqcup K\gamma$, $F\mu \sqcap H\nu \sqcap K\gamma = \tilde{0}\theta$, $F\mu \sqcap H\nu \neq \tilde{0}\theta$ and $F\mu \sqcap K\gamma \neq \tilde{0}\theta$. Then, $F\mu = M\psi \sqcup N\eta$ where $M\psi = F\mu \sqcap H\nu \sqsubseteq H\nu$ and $N\eta = F\mu \sqcap K\gamma \sqsubseteq K\gamma$. Since $F\mu \sqcap H\nu \sqcap K\gamma = \tilde{0}\theta$ and $M\psi \sqsubseteq H\nu$, then $F\mu \sqcap M\psi \sqcap K\gamma = \tilde{0}\theta$. Also, since $M\psi \sqsubseteq F\mu$, then $M\psi \sqcap K\gamma = \tilde{0}\theta$. Therefore, $M\psi q K\gamma$, Similarly, $N\eta q H\nu$. Hence, $F\mu$ is not a GFS weakly-connected. This complete the proof.

Theorem 3.14

A GFS weakly-connected set in $(X,)$ is $GFSC3$-connected.

Proof. Let $F\mu$ be a GFS weakly-connected set in $(X,)$. Suppose $F\mu$ is not $GFSC3$-connected. Then, there exist $H\nu$ and $K\gamma \in T$ such that $F\mu \sqsubseteq H\nu \sqcup K\gamma$, $H\nu \sqcap K\gamma \sqsubseteq F\mu c$, $H\nu \not\sqsubseteq F\mu c$ and $K\gamma \not\sqsubseteq F\mu c$. Then, $F\mu = M\psi \sqcup N\eta$ where $M\psi = F\mu \sqcap H\nu \sqsubseteq H\nu$ and $N\eta = F\mu \sqcap K\gamma \sqsubseteq K\gamma$. Let $J\sigma$ and $L\rho \in G(X, E)$ defined by: $J\sigma = \{M\psi,\ H\nu \sqsupseteq K\gamma, \tilde{0}\theta$, otherwise $L\rho = \{N\eta,\ K\gamma \sqsupset H\nu, \tilde{0}\theta$, otherwise. Then $F\mu = J\sigma \sqcup L\rho$.
Now, $(e)(x) \neq 0$, $\sigma(e) \neq 0$. For, $(e)(x) = 0$, $\sigma(e) = 0$. Since, $H\nu \not\sqsubseteq F\mu c$, then there exist $x \in X, e \in E$ such that $H(e)(x) + F(e)(x) > 1$, $\nu(e) + \mu(e) > 1$. Then, $(e)(x) > K(e)(x)$, $\nu(e) > \gamma(e)$. For, $H(e)(x) \leq K(e)(x)$, $\nu(e) \leq \gamma(e)$ implies $K(e)(x) + F(e)(x) > 1$, $\gamma(e) + \mu(e) > 1$ and hence $(H\nu \sqcap K\gamma)(e)(x) > 1 - F\mu(e)(x)$ i.e., $H(e)(x) > 1 - F(e)(x)$, $\nu(e) > 1 - \mu(e)$ and $K(e)(x) > 1 - F(e)(x)$, $\gamma(e) > 1 - \mu(e)$ this is a contradiction with $H\nu \sqcap K\gamma \sqsubseteq F\mu c$. So, $(e)(x) \neq 0$, $\sigma(e) \neq 0$. Similarly, $(e)(x) \neq 0$, $\rho(e) \neq 0$. Also, $J\sigma \sqsubseteq M\psi \sqsubseteq H\nu$ and $L\rho \sqsubseteq N\eta \sqsubseteq K\gamma$. Now, $J\sigma q K\gamma$. For, if $J\sigma q K\gamma$, then there exist $x \in X, e \in E$ such that $J(e)(x) + K(e)(x) > 1$, $\sigma(e) + \gamma(e) > 1$ and hence $J(e)(x) > 0$, $\sigma(e) > 0$. This means $H(e)(x) \geq K(e)(x)$, $\nu(e) \leq \gamma(e)$ and so $F(e)(x) = M(e)(x)$, $\mu(e) = \psi(e)$ implying $F(e)(x) + H(e)(x) > 1$, $\mu(e) + \nu(e) > 1$ and thus $(H\nu \sqcap K\gamma)(e)(x) > 1 - F\mu(e)(x)$ which is a contradiction with $H\nu \sqcap K\gamma \sqsubseteq F\mu c$. Similarly, $L\rho q H\nu$. Thus, $J\sigma$ and $L\rho$ are GFS weakly separated and $F\mu = J\sigma \sqcup L\rho$. So, $F\mu$ is not a GFS weakly-connected. This a contradiction. Then $F\mu$ is a $GFSC3$-connected.

Remark 3.15

The $GFSC3$-connected set (respectively, $GFSC2$-connected) may not be a GFS weakly-connected.

Theorem 3.16

The $GFSC3$-connected set in $(X, .)$ is a GFS Q-connected.

Proof. Let $F\mu$ be a $GFSC3$-connected set in $(X,.)$. Suppose $F\mu$ is not GFS Q-connected. Then, there exist two non-null GFS Q-separated sets $H\nu$ and $K\gamma$ such that $F\mu = H\nu \sqcup K\gamma$, $(H\nu) \sqcap K\gamma = H\nu \sqcap cl(K\gamma) = \tilde{0}\theta$. This implies that $K\gamma \sqsubseteq [cl(H\nu)]c$ and $H\nu \sqsubseteq [cl(K\gamma)]c$. Let $M\psi = [cl(H\nu)]c$ and $N\eta = [cl(K\gamma)]c$. Then, $M\psi$ and $N\eta$ are non- null GFS open sets such that $F\mu \sqsubseteq M\psi \sqcup N\eta$. Now, $M\psi \sqcap N\eta = [cl(H\nu)]c \sqcap [cl(K\gamma)]c = [cl(H\nu) \sqcup cl(K\gamma)]$

$c=[cl(H\nu \sqcup K\gamma)]c \sqsubseteq F\mu c$. Also, $M\psi \not\sqsubseteq F\mu c$. For, if $M\psi \sqsubseteq F\mu c$, then $F\mu \sqsubseteq M\psi c=(H\nu)$ which would imply $K\gamma=\tilde{0}\theta$ (since $cl(H\nu)\sqcap K\gamma=\tilde{0}\theta$). This is a contradiction. Similarly, $N\eta \not\sqsubseteq F\mu c$. Therefore, $F\mu$ is not $GFSC3$-connected. So, $F\mu$ is GFS Q-connected.

Theorem 3.17

A $GFSS$ $F\mu$ in (X, .) is $GFSC2$-connected if and only if $F\mu$ is GFS s-connected.

Proof. Let $F\mu$ be a $GFSC2$-connected set in (X,). Suppose $F\mu$ is not a GFS s-connected. Then there exist non-null GFS separated sets $H\nu$ and $K\gamma$ in (X,) such that $F\mu=H\nu \sqcup K\gamma$. Then, there exist two non- null GFS open sets $M\psi$ and $N\eta$ such that $H\nu \sqsubseteq M\psi$, $K\gamma \sqsubseteq N\eta$, and $H\nu \sqcap N\eta=K\gamma \sqcap M\psi=\tilde{0}\theta$. Then, $F\mu \sqsubseteq M\psi \sqcup N\eta$. Now, $F\mu \sqcap M\psi \sqcap N\eta=(H\nu \sqcup K\gamma)\sqcap M\psi \sqcap N\eta=(H\nu \sqcap M\psi \sqcap N\eta)\sqcup(K\gamma \sqcap M\psi \sqcap N\eta)=\tilde{0}\theta$ and $F\mu \sqcap M\psi=(H\nu \sqcup K\gamma)\sqcap M\psi=(H\nu \sqcap M\psi)\sqcup(K\gamma \sqcap M\psi)=H\nu \neq \tilde{0}\theta$. Similarly, $F\mu \sqcap N\eta \neq \tilde{0}\theta$. So, $F\mu$ is not $GFSC2-$connected which is a contradiction. Conversely, let $F\mu$ be GFS s-connected. Suppose that $F\mu$ is not $GFSC2$-connected. Then there exist two non-null GFS open sets $M\psi$ and $N\eta$ such that $F\mu \sqsubseteq M\psi \sqcup N\eta$, $F\mu \sqcap M\psi \sqcap N\eta=\tilde{0}\theta$, $F\mu \sqcap M\psi \neq \tilde{0}\theta$, $F\mu \sqcap N\eta \neq \tilde{0}\theta$. Hence, $F\mu=H\nu \sqcup K\gamma$ where $H\nu=F\mu \sqcap M\psi \sqsubseteq M\psi$ and $K\gamma=F\mu \sqcap N\eta \sqsubseteq N\eta$. Also, $K\gamma \sqcap M\psi=(F\mu \sqcap N\eta)\sqcap M\psi=\tilde{0}\theta$, Similarly, $H\nu \sqcap N\eta=\tilde{0}\theta$. So, $F\mu$ is not GFS s-connected and this complete the proof.

Theorem 3.18

The $GFSC4$-connected set in (X, .) is a GFS strongly-connected.

Proof. Let $F\mu$ be a $GFSC4$-connected set in (X,.). Suppose $F\mu$ is not a GFS strongly-connected. Then there exist two non-null GFS strongly separated sets $H\nu$ and $K\gamma$ in (X,) such that $F\mu=H\nu \sqcup K\gamma$. So, there exist two non- null GFS open sets $M\psi$ and $N\eta$ such that $H\nu \sqsubseteq M\psi$, $K\gamma \sqsubseteq N\eta$, and $H\nu \sqcap N\eta=K\gamma \sqcap M\psi=\tilde{0}\theta$, $H\nu$ and $M\psi$ GFS quasi-coincident with respect to $H\nu$, and $K\gamma$ and $N\eta$ GFS quasi-coincident with respect to $K\gamma$. Then, for every $x,e \in S(H\nu)$ we have $H(e)(x)+M(e)(x)>1$ and $\nu(e)+\psi(e)>1$ and for every $x,e \in S(K\gamma)$ we have $K(e)(x)+N(e)(x)>1$ and $\gamma(e)+\eta(e)>1$. Then, $F\mu \sqsubseteq M\psi \sqcup N\eta$. Also, $F\mu \sqcap M\psi \sqcap N\eta=\tilde{0}\theta$. Again, $F(e)(x)+M(e)(x)>H(e)(x)+M(e)(x)$ and $\mu(e)+\psi(e)>\nu(e)+\psi(e)>$ for every $x,e \in S(H\nu)$. Therefore, $M\psi \not\sqsubseteq F\mu c$, Similarly, $N\eta \not\sqsubseteq F\mu c$. Thus, $F\mu$ is not a $GFSC4-$connected. This is a contradiction. So, $F\mu$ is a GFS strongly-connected.

Theorem 3.19

Let (X,$T1$,E) and (Y,$T2$,K) be a $GFST$-spaces and fup:(X,$T1$,E)⟶(Y,$T1$,K) be a GFS-continuous bijective mapping. If $F\mu$ is a $GFSCi$-connected (respectively, GFS s-connected, GFS strongly-connected, GFS weakly-connected, GFS clopen-connected) set in (X,) for i=1, 2, then $fu(F\mu)$ is a $GFSCi$-connected (respectively, GFS s-connected, GFS strongly-connected, GFS weakly-connected, GFS clopen-connected) set in (Y,K) for i=1, 2.

Proof. The case of $GFSCi$-connected set **(**i=1,2) previously proved (see [11]). Now, we prove the case of GFS clopen-connected. Let $F\mu$ be a GFS-clopen connected set in (X, .). Suppose $f(F\mu)$ is not a GFS clopen-connected set in (Y,K). Then, $f(F\mu)$ has non-null proper clopen GFS subset of $J\sigma$. So, there exist $S\varepsilon \in T2$ and $L\rho \in T2c$ such that $J\sigma=f(F\mu)\sqcap S\varepsilon=fup(F\mu)\sqcap L\rho$. Since, fup is injective mapping, then $fup^{-1}(J\sigma)=F\mu \sqcap fup^{-1}(S\varepsilon)=F\mu \sqcap fup^{-1}(L\rho)$. Also, since $S\varepsilon \in T2$ and $L\rho \in T2c$ and fup is a GFS-continuous mapping, then $fup^{-1}(S\varepsilon)\in T1$ and $fup^{-1}(L\rho)\in T1c$. Hence, $fup^{-1}(J\sigma)$ is non-null proper clopen GFS subset of $F\mu$ which is a contradiction. Therefore, $f(F\mu)$ is a GFS-clopen connected set in (Y,K). The cases of $GFSC3$-connected and $GFSC4$-connected sets we need to the GFS-continuous surjective mapping previously proved (see [11]).

Theorem 3.20

Let (X,$T1$,E) and (Y,$T2$,K) be a $GFST$-spaces and fup:(X,$T1$,E)⟶(Y,$T1$,K) be a GFS injective mapping. If $F\mu$ is a GFS $Q-$connected set in (X,), then $fu(F\mu)$ is a GFS Q-connected set in (Y,K).

Proof. Let $F\mu$ be a GFS $Q-$connected set in (X,). Suppose $f(F\mu)$ is not a GFS $Q-$connected set in (Y,K). Then, there exist two non- null GFS Q separated sets $J\sigma$ and $L\rho$ in (X,) such that $f(F\mu)$ $=J\sigma \sqcup L\rho$, $cl(J\sigma)\sqcap L\rho=J\sigma \sqcap cl(L\rho)=\tilde{0}\theta Y$. Since, fup is injective mapping, then $fup^{-1}(fup(F\mu)$) $=fup^{-1}(J\sigma)\sqcup fup^{-1}(L\rho)$, This means that, $fup^{-1}(J\sigma)$, $fup^{-1}(L\rho)$ are GFS Q separated sets of $F\mu$ in (X,E), which is contradicts of the GFS Q-connectedness of $F\mu$ in (X,E).

$cl(fup^{-1}(J\sigma))\sqcap fup^{-1}(L\rho)\sqsubseteq fup^{-1}(cl(J\sigma))\sqcap fup^{-1}(L\rho)=fup^{-1}(cl(J\sigma)\sqcap L\rho))=fup^{-1}(\tilde{0}\theta Y)=\tilde{0}\theta X$, $fup^{-1}(J\sigma)\sqcap cl(fup^{-1}(L\rho))\sqsubseteq fup^{-1}(J\sigma\sqcap fup^{-1}(cl(L\rho))=fup^{-1}(L\rho\sqcap cl(L\rho))=fup^{-1}(\tilde{0}\theta Y)=\tilde{0}\theta X$. Therefore, $f(F\mu)$ is a GFS Q-connected set in (Y,K).

Theorem 3.21

Let $(X,T1,E)$ and $(Y,T2,K)$ be a $GFST$-spaces and $fup:(X,T1,E)\longrightarrow(Y,T1,K)$ be a GFS-bijective open mapping. If $G\delta$ is a $GFSCi$-connected(respectively, GFS s-connected, GFS strongly-connected, GFS Q-connected, GFS weakly-connected, GFS clopen-connected) set in (Y,E) for i=1,2,3,4, then $fup^{-1}(G\delta)$ is a $GFSCi$-connected (respectively, GFS s-connected, GFS strongly-connected, GFS Q-connected,GFS weakly-connected, GFS-clopen connected) set in (Y,E) for i=1,2,3,4.

Proof. The case of $GFSCi$-connected set **(**i=1,2,3,4) previously proved (see [13]). Now, we will prove the case of GFS s-connected. Let $G\delta$ is a GFS $s-$connected set in $(Y,)$. Suppose $fup^{-1}(G\delta)$ is not a GFS s-connected set in (X,E). Then, there exist two non- null GFS separated sets $H\nu$ and $K\gamma$ in $(X,)$ such that $fup^{-1}(G\delta)=H\nu\sqcup K\gamma$. Therefore, there exist two non- null GFS open sets $M\psi$ and $N\eta$ in $(X,)$ such that $H\nu\sqsubseteq M\psi$ and $K\gamma\sqsubseteq N\eta$ and $H\nu\sqcap N\eta=K\gamma\sqcap M\psi=\tilde{0}$. Since, fup is a GFS surjective mapping, then $f(fup^{-1}(G\delta))=G\delta$ and so $G\delta=fup(H\nu\sqcup K\gamma)=fup(H\nu)\sqcup fup(K\gamma)$. Since, fup is a GFS open mapping, then $f(M\psi)$ and $fup(N\eta)$ are non-null GFS open sets in (Y,K) such that $fup(H\nu)\sqsubseteq fup(M\psi)$, $fup(K\gamma)\sqsubseteq fup(N\eta)$. Since, fup is a GFS injective mapping, then $fup(H\nu)\sqcap fup(N\eta)=fup(H\nu\sqcap N\eta)=\tilde{0}\theta Y$ and $fup(K\gamma)\sqcap fup(M\psi)=\tilde{0}\theta Y$. It follows that $G\delta$ is not a GFS s-connected set, a contradiction.

Theorem 3.22

If $F\mu$ and $G\delta$ are intersecting $GFSC1$-(respectively, $GFSC2$-connected, GFS s=connected, GFS weakly-connected, GFS Q-connected, GFS strongly-connected) sets in $(X,)$. Then, $F\mu\sqcup G\delta$ is a $GFSC1$-connected (respectively, $GFSC2$-connected, GFS s-connected, GFS weakly-connected, GFS Q-connected, GFS strongly-connected) set in $(X, .)$.

Proof. The cases of $GFSC1$-connected and $GFSC2$connected sets is previously proved (see [14]). Now, we will prove the case of GFS Q-connected sets. Let $F\mu$ and $G\delta$ are intersecting GFS Q-connected sets in $(X,)$. Suppose$F\mu\sqcup G\delta$ is not a GFS Q-connected set. Then, there exist two non- null GFS $Q-$separated sets $H\nu$ and $K\gamma$ in $(X,)$ such that $F\mu\sqcup G\delta=H\nu\sqcup K\gamma$. Therefore, $F\mu\sqcap H\nu$, $F\mu\sqcap K\gamma$, $G\delta\sqcap H\nu$ and $G\delta\sqcap K\gamma$ are non-null GFS $Q-$separated sets in $(X,)$ as subsets of $H\nu$ and $K\gamma$. Since, $F\mu=(F\mu\sqcap H\nu)\sqcup(F\mu\sqcap K\gamma)$ and $G\delta=(F\mu\sqcap H\nu)\sqcup(F\mu\sqcap K\gamma)$, then $F\mu$ and $G\delta$ are not GFS $Q-$connected which is a contradiction.

Theorem 3.23

Let $\{(F\mu):i\in J\}$ be a family of a $GFSC1$-connected (respectively, $GFSC2$-connected, GFS s-connected, GFS weakly-connected, GFS Q-connected, GFS strongly-connected) sets in $(X,)$ such that for $i,j\in J$, the $GFSSs$ $(F\mu)i$ and $(F\mu)j$ are intersecting. Then, $F\mu=\bigsqcup i\in(F\mu)i$ is a $GFSC1$-connected (respectively, $GFSC2$-connected, GFS s-connected, GFS weakly-connected, GFS Q-connected, GFS strongly-connected) set in (X,E).

Proof. The case of $GFSC1$-connected set previously proved (see [5]). Now, we will prove the case of $GFSC2$-connected set. Let $\{(F\mu):i\in J\}$ be family of $GFSC2$-connected sets in $(X,)$. Suppose that $F\mu$ is not a $GFSC2$-connected set in $(X,)$. Then, there exist two GFS open sets $H\nu$ and $K\gamma$ in $(X,)$ such that $F\mu\sqsubseteq H\nu\sqcup K\gamma$, $F\mu\sqcap H\nu\sqcap K\gamma=\tilde{0}\theta$, $F\mu\sqcap H\nu\neq\tilde{0}\theta$ and $F\mu\sqcap K\gamma\neq\tilde{0}\theta$. Now, let $(F\mu)0$ be any $GFSS$ of the given family. Then, $(F\mu)0\sqsubseteq H\nu\sqcup K\gamma$, $H\nu\sqcap K\gamma\sqsubseteq(F\mu)i0c$. But, $(F\mu)0$ is a $GFSC2$-connected set. Hence, $(F\mu)0\sqcap H\nu=\tilde{0}\theta$ or $(F\mu)i0\sqcap K\gamma=\tilde{0}\theta$. Now if $(F\mu)0\sqcap H\nu=\tilde{0}\theta$, we can prove that $(F\mu)i\sqcap H\nu=\tilde{0}\theta$ for each $i\in J-\{i0\}$ and so $F\mu\sqcap H\nu=\tilde{0}\theta$. This complete the proof.

Corollary 3.24

If $\{(F\mu):i\in J\}$ is a family of a $GFSC1$-connected (respectively, $GFSC2$-connected, GFS s-connected, GFS weakly-connected, GFS Q-connected, GFS strongly-connected) sets in X and $\sqcap i\in J(F\mu)i\neq\tilde{0}\theta$, then $F\mu=\bigsqcup i\in J(F\mu)i$ is a $GFSC1$-connected (respectively, $GFSC2$-connected, GFS s-connected, GFS weakly-connected, GFS Q-connected, GFS strongly-connected) set in (X,E).

Theorem 3.25

If $F\mu$ and $G\delta$ are GFS quasi-coincident $GFSC3$-connected (respectively, $GFSC4$-connected) sets in $(X,)$, then $F\mu\sqcup G\delta$ is a $GFSC3$-

connected (respectively, $GFSC4$-connected) set in (X,E).

Proof. As a sample, we will prove the case $GFSC3$−connected. Let $F\mu$ and $G\delta$ be GFS quasi-coincident $GFSC3$−connected sets in $(X,)$. Suppose there exist two non-null GFS open sets $H\nu$ and $K\gamma$ in $(X,)$ such that $F\mu \sqcup G\delta \sqsubseteq H\nu \sqcup K\gamma$ and $H\nu \sqcap K\gamma \sqsubseteq (F\mu \sqcup G\delta)c$. (1) [we prove that $H\nu \sqsubseteq (F\mu \sqcup G\delta)c$ or $K\gamma \sqsubseteq (F\mu \sqcup G\delta)c$] Therefore, $F\mu \sqsubseteq H\nu \sqcup K\gamma$, $H\nu \sqcap K\gamma \sqsubseteq F\mu c$, $G\delta \sqsubseteq H\nu \sqcup K\gamma$ and $H\nu \sqcap K\gamma \sqsubseteq G\delta c$. Since, $F\mu$ and $G\delta$ are $GFSC3$−connected, then ($H\nu \sqsubseteq F\mu c$ or $K\gamma \sqsubseteq F\mu c$) and ($H\nu \sqsubseteq G\delta c$ or $K\gamma \sqsubseteq G\delta c$). Moreover, since $F\mu$ and $G\delta$ are GFS quasi-coincident, there exist $x \in X, \in E$ such that
$(e)(x) > 1-(e)(x)$ and $\mu(e) > 1-\delta(e)$. (2) Now, consider the following cases:
Case 1. Suppose $H\nu \sqsubseteq F\mu c$. Then, by (2) we have, $1-H(e)(x) \geq F(e)(x) > 1-G(e)(x)$ and $1-\nu(e) \geq \mu(e) > 1-\delta(e) \Rightarrow H(e)(x) < G(e)(x)$ and $\nu(e) < \delta(e)$. (3) We claim that, $K\gamma \not\sqsubseteq G\delta c$. For if not, then $K(e)(x) \leq 1-G(e)(x) < F(e)(x)$ and $\gamma(e) \leq 1-\delta(e) < \mu(e)$. (4) Now by (3) and (4), we have $H(e)(x) \vee K(e)(x) < F(e)(x) \vee G(e)(x)$ and $\nu(e) \vee \gamma(e) < \mu(e) \vee \delta(e)$ which implies $F\mu \sqcup G\delta \not\sqsubseteq H\nu \sqcup K\gamma$, this contradicts (1). Hence, $H\nu \sqsubseteq G\delta c$. Therefore, $H\nu \sqsubseteq F\mu c \sqcap G\delta c = (F\mu \sqcup G\delta)c$.
Case 2. Suppose $K\gamma \sqsubseteq F\mu c$. Here, we can show as in Case 1 that $H\nu \not\sqsubseteq G\delta c$. Therefore, $K\gamma \sqsubseteq G\delta c$. Hence, $K\gamma \sqsubseteq G\delta c$. Therefore, $K\gamma \sqsubseteq F\mu c \sqcap G\delta c = (F\mu \sqcup G\delta)c$. This complete the proof.

Theorem 3.26

Let $\{(F\mu): i \in J\}$ be a family of $GFSC3$-connected (respectively, $GFSC4$-connected,) sets in $(X,)$ such that for $i,j \in J$, the $GFSSs$ $(F\mu)i$ and $(F\mu)j$ are GFS quasi-coincident. Then, $F\mu = \bigsqcup i \in (F\mu)i$ is a $GFSC3$-connected (respectively, $GFSC4$−connected) set in (X,E).

Proof. Let $\{(F\mu): i \in J\}$ be family of $GFSC3$-connected sets in $(X,)$. Suppose there exist two GFS open sets $H\nu$ and $K\gamma$ in $(X,)$ such that $F\mu \sqsubseteq H\nu \sqcup K\gamma$ and $H\nu \sqcap K\gamma \sqsubseteq F\mu c$. Let $(F\mu)0$ be any $GFSS$ of the given family. Then, $(F\mu)0 \sqsubseteq H\nu \sqcup K\gamma$, $H\nu \sqcap K\gamma \sqsubseteq (F\mu)i0c$. Since, $(F\mu)0$ is a $GFSC3$-connected set, we have $H\nu \sqsubseteq (F\mu)i0c$ or $K\gamma \sqsubseteq (F\mu)i0c$. Now, the result follows in view of the facts that $(F\mu)i0 \sqsubseteq H\nu c$, then $(F\mu)i \sqsubseteq H\nu c$ for each $i \in J-\{i0\}$, since $(F\mu)i0$ and $(F\mu)i$ are GFS quasi-coincident $GFSC3$−connected sets, and $H\nu \sqsubseteq [\sqcap i \in J(F\mu)i]c = F\mu c$. Hence, $F\mu$ is a $GFSC3$-connected. Similarly, if $\{(F\mu): i \in J\}$ is family of $GFSC4$-connected sets in $(X,)$ such that for $i,j \in J$, the $GFSSs$ $(F\mu)i$ and $(F\mu)j$ are GFS quasi-coincident, then, $F\mu = \bigsqcup i \in J(F\mu)i$ is a $GFSC4$-connected set in (X,E). This completes the proof.

Corollary 3.27

Let $\{(F\mu): i \in J\}$ be a family of a $GFSC3$-connected (respectively, $GFSC4$-connected,) sets in $(X,)$ and $(x\alpha, e\lambda)$ be a GFS point such that $\alpha > 12$, $\lambda > 12$ and $(x\alpha, e\lambda) \in \sqcap i \in J(F\mu)i$. Then $\bigsqcup i \in (F\mu)i$ is a $GFSC3$-connected (respectively, $GFSC4$−connected) set in (X,E).

Proof. Since $(x\alpha,) \in \sqcap i \in J(F\mu)i$, then $(x\alpha, e\lambda) \in (F\mu)i$ for each $i \in J$. Therefore, $(F\mu)$ and $(F\mu)$ are GFS quasi-coincident for each $i,j \in J$. By Theorem 4.13, $\bigsqcup i \in (F\mu)i$ is a $GFSC3$-connected (respectively, $GFSC4$-connected) set in (X,E).

Theorem 3.28

If $F\mu$ is a $GFSC3$-connected (respectively, $GFSC4$-connected, GFS strongly-connected, GFS Q-connected) set in $(X,)$ and $F\mu \sqsubseteq G\delta \sqsubseteq c(F\mu)$, then $G\delta$ is also a $GFSC3$-connected (respectively, $GFSC4$-connected, GFS strongly-connected, GFS Q-connected) set in (X,E). In particular $(F\mu)$ is $GFSC3$-connected (respectively, $GFSC4$-connected, GFS strongly-connected, GFS Q-connected) set in (X,E).

Proof. As a sample, we will prove the case $GFSC3$−connected. Let $H\nu$ and $K\gamma$ be GFS open sets in $(X,)$ such that $G\delta \sqsubseteq H\nu \sqcup K\gamma$ and $H\nu \sqcap K\gamma \sqsubseteq G\delta c$. Then, $F\mu \sqsubseteq H\nu \sqcup K\gamma$ and $H\nu \sqcap K\gamma \sqsubseteq F\mu c$. Since $F\mu$ is a $GFSC3$−connected set, we have $F\mu \sqsubseteq H\nu c$ or $F\mu \sqsubseteq K\gamma c$. But, if $F\mu \sqsubseteq H\nu c$, then $(F\mu) \sqsubseteq H\nu c$ and on the other hand, if $F\mu \sqsubseteq K\gamma c$, then $cl(F\mu) \sqsubseteq K\gamma c$. Therefore, $G\delta \sqsubseteq (F\mu) \sqsubseteq H\nu c$ or $G\delta \sqsubseteq cl(F\mu) \sqsubseteq K\gamma c$. Hence, $G\delta$ is a $GFSC3$−connected set in $(X,)$. This completes the proof.

Conclusions

In the present paper have extended the notion of connectedness of fuzzy soft topological spaces to generalized fuzzy soft topological spaces. We have introduced different notions of generalized fuzzy soft separated sets and studied the relationship between them. The study has also been devoted to introduce the different notions of connectedness in generalized fuzzy soft topological spaces and study the implications that exist between them. However, we note that the last theorem above fails in case of $GFSC1$-connectedness (respectively, $GFSC2$-

connectedness, GFS clopen-connectedness, GFS weakly-connectedness, GFS s-connectedness) which is a departure from general topology. In fact, closure of a $GFSC1$-connected (respectively, $GFSC2$-connected, GFS clopen-connected, GFS weakly-connected, GFS s-connected) set need not be a $GFSC1$-connected (respectively, $GFSC2$-connected, GFS clopen-connected, GFS weakly-connected, GFS s-connected).

Conflicts of interest

The authors declare no conflict of interest.

References

[1] Andrijevic D. On b-open sets. Mat Vesnik 1996;48: 9-64.

[2] Brooks F. Indefinite cut sets for real functions. Amer Math Monthly 1971;78: 1007-10.

[3] Change CL. Fuzzy topological spaces. J Math Anal Appl 1968;24:182-190.

[4] Chinnadurai V, Bharathivelan K. Cubic Ideals in Near Subtraction Semigroups. Int J Mod Sci Technol 2016;1(8):276-82.

[5] Davis A. Indexed systems of neighborhoods for general topological spaces. Amer Math Monthly 1961;68(9):886-93.

[6] Dunham W. Weakly Hausdorff spaces. Kyungpook Math J 1975;15(1):41-50.

[7] Ekici E. On contra-continuity. Ann Univ Sci Bodapest 2004;47:127-37.

[8] Ekici E. New forms of contra-continuity. Carpathian J Math 2008;24(1):37-45.

[9] Katˇetov M. On real-valued functions in topological spaces. Fund Math 1951;38: 85-91.

[10] Lane E. Insertion of a continuous function. Pacific J Math 1976;66:181-90.

[11] Maheshwari SN, Prasad R. On ROs-spaces. Portugal Math 1975;34:213-217.

[12] Maji PK, Biswas R, Roy PR. Fuzzy soft sets. J Fuzzy Math 2009;9:589-602.

[13] Okelo NB. On characterization of various finite subgroups of Abelian groups. Int J Mod Comp Info and Com Technol 2018;1(5):93-8.

[14] Saha S. Local connectedness in fuzzy setting. Simon Stevin 1987;61:3-13.

[15] Shabir M, NazM. On soft topological spaces. Comput Math Appl 2011;61:1786-99.

[16] Stone M. Application of Boolean algebras to topology. Mat Sb 1936; 1: 765-771.

[17] Vijayabalaji S, Sathiyaseelan N. IntervalValued Product Fuzzy Soft Matrices and its Application in Decision Making. Int J Mod Sci Technol 2016;1(7):159-63.

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