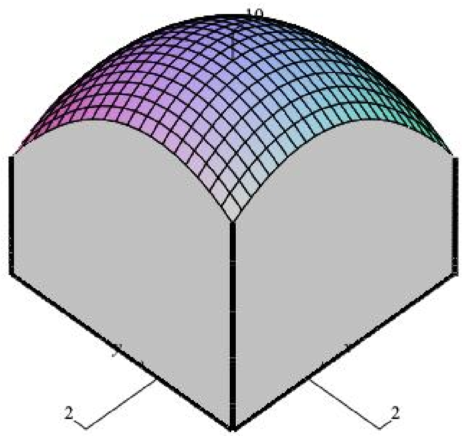
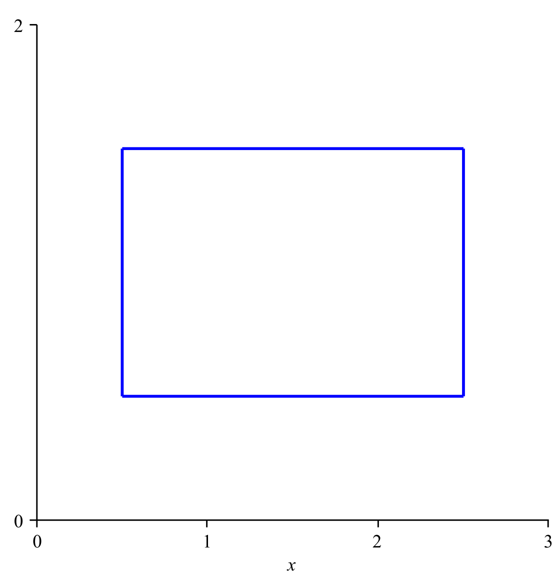
Calculus 3 - Double Integrals

Last class we consider the problem of finding the volume under $z = f(x, y)$ on the rectangular region $[a, b] \times [c, d]$



were we introduced the double integral

$$\int_a^b \int_c^d f(x, y)\, dydx \tag{1}$$



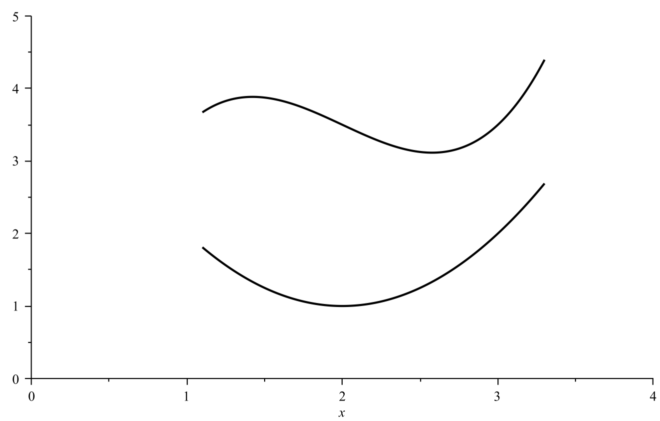
noting that to evaluate the integral, we first hold one variable fixed and integrate with respect to the other and in this case hold x fixed,

$$\int_a^b \int_c^d f(x,y)\,dydx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx \tag{2}$$

and then we integral with respect to y.

In general we have

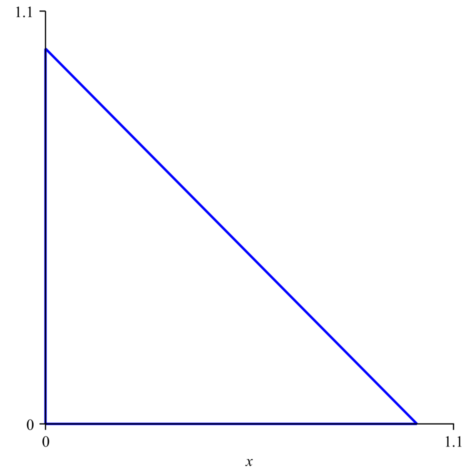
$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)dydx \tag{3}$$



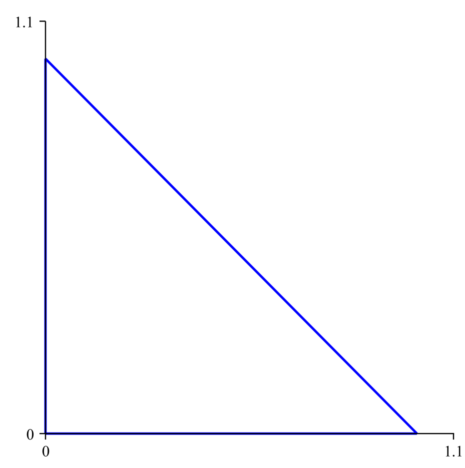
As an example, let us set up and evaluate the integral

$$\iint\limits_R 12x^2y\,dA \tag{4}$$

where $dA = dydx$ and where R is the region bound by $x + y = 1$, $x = 0$ and $y = 0$. We first sketch region



$$
\begin{aligned}
V &= \int_0^1 \int_0^{1-x} 12x^2 y dy dx \\
&= \int_0^1 6x^2y^2 \Big|_0^{1-x} \, dx \\
&= \int_0^1 6x^2(1-x)^2 \, dx \\
&= \int_0^1 6x^2 - 12x^3 + 6x^4 \, dx \\
&= 2x^3 - 3x^4 + \frac{6}{5}x^5 \ \Big|_0^1 \\
&= 2 - 3 + \frac{6}{5} = \frac{1}{5}
\end{aligned} \tag{5}
$$



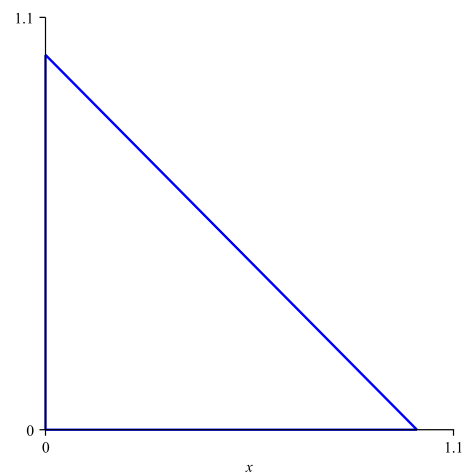
If we want to integration with respect to x first

$$V = \int_?^? \int_?^? 12x^2y \; dxdy \tag{6}$$

we need to determine the limits. With vertical rectangles we went from a bottom curve to top curve and in the last example

$$y = 0 \quad \rightarrow \quad y = 1 - x \tag{7}$$

so integrating with respect to x we use horizontal rectangles and go from



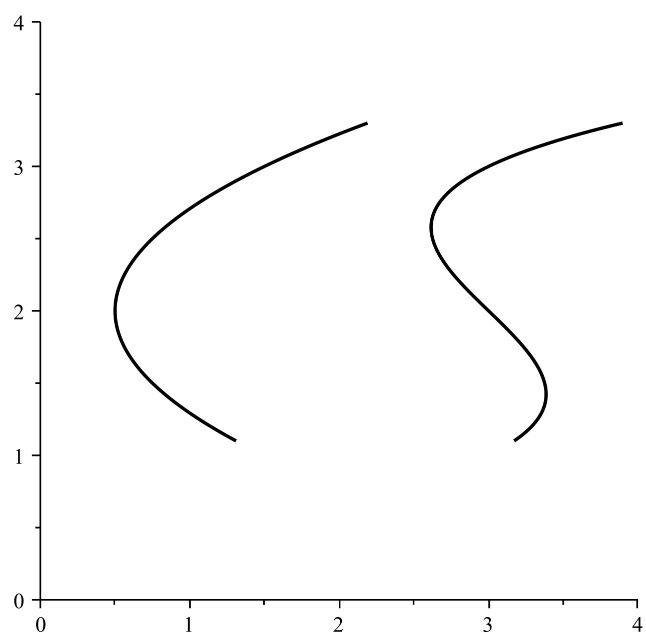
a left curve to a right curve and so

$$x = 0 \quad \rightarrow \quad x = 1 - y \tag{8}$$

and so

$$
\begin{aligned}
V &= \int_0^1 \int_0^{1-y} 12x^2 y dx dy \\
&= \int_0^1 4x^3 y \Big|_0^{1-y} dx \\
&= \int_0^1 4(1-y)^3 y \, dx \\
&= \int_0^1 4(1-3y+3y^2-y^3) y \, dx \\
&= \int_0^1 4y - 12y^2 + 12y^3 - 4y^4 dy \\
&= 2y^2 - 4y^3 + 3y^4 - \frac{4}{5} y^5 \Big|_0^1 \\
&= 2 - 4 + 3 - \frac{4}{5} = \frac{1}{5}
\end{aligned}
\tag{9}
$$

In general we have



$$
\int_{y=c}^{y=d} \int_{x=G(y)}^{x=H(y)} f(x,y) dx dy \tag{10}
$$

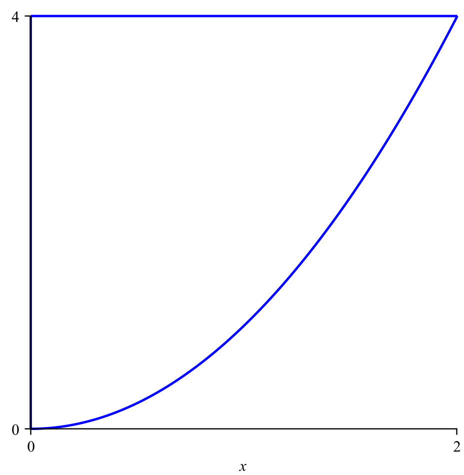
or simply

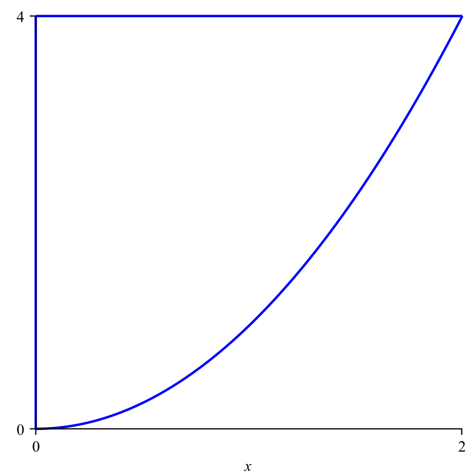
$$\int_c^d \int_{G(y)}^{H(y)} f(x,y)dxdy \tag{11}$$

Example 2. Set up and evaluate

$$\iint_R \frac{y}{x+1} dA \tag{12}$$

where R is the region bound by $y = x^2$, $y = 4$ and $x = 0$. We first sketch the region so





$$\int_0^2 \int_{x^2}^4 \frac{y}{x+1} dydx \qquad \int_0^4 \int_0^{\sqrt{y}} \frac{y}{x+1} dxdy \tag{13}$$

Which is easier to integrate?

$$\frac{1}{2}\int_0^2 \left. \frac{y^2}{x+1} \right|_0^{x^2} dx \qquad \int_0^4 y \ln(x+1) \Big|_0^{\sqrt{y}} dy \tag{14}$$

then

$$= \frac{1}{2}\int_0^2 \frac{x^4}{x+1} dx \qquad = \int_0^4 y \ln(\sqrt{y}+1) dy \tag{15}$$

Reversing the Order of Integration

Sometimes we are given the integral already with limits and are asked to change or reverse the order of integration. For example, suppose we are given (pg 977, # 54)

$$\int_0^9 \int_{\sqrt{x}}^3 dydx \tag{16}$$

and are asked to create a double integral

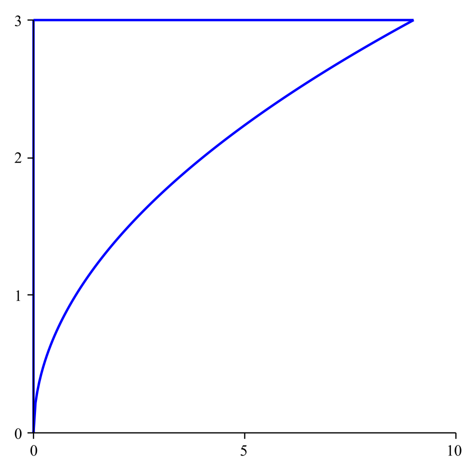
$$\int_?^? \int_?^? dxdy \tag{17}$$

Since the inside integral is with respect to y so

$$y = \sqrt{x} \quad \rightarrow \quad y = 4. \tag{18}$$

These two curves we can draw. The outside integral is point to point in the x and so

$$x = 0 \rightarrow x = 9. \tag{19}$$



$$\int_0^3 \int_0^{y^2} dxdy \tag{20}$$

Sometimes it is necessary to reverse the order of integration to actual integrate. The following demonstrates (pg 977, # 64). Integrate the following:

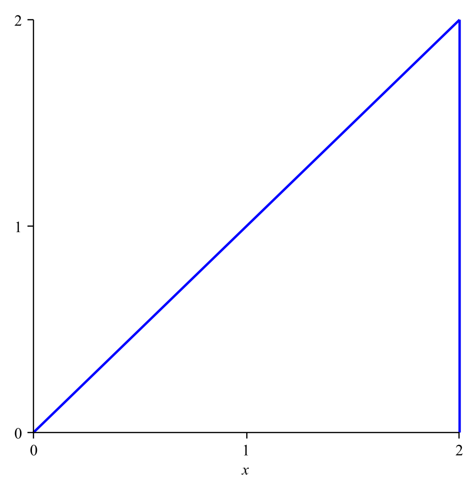
$$\int_0^2 \int_y^2 e^{-x^2} dx dy. \tag{21}$$

As it stand, we can't right now so we will reverse of the order of integration. Since the inside integral is with respect to x (left curve to right curve) so

$$x = y \quad \rightarrow \quad x = 2. \tag{22}$$

These two curves we can draw. The outside integral is point to point in the y direction and so

$$y = 0 \quad \rightarrow \quad y = 2. \tag{23}$$



$$\begin{aligned} \int_0^2 \int_0^x e^{-x^2} dy dx &= \int_0^2 e^{-x^2} y \Big|_0^x dx \\ &= \int_0^2 x e^{-x^2} dx \\ &= -\frac{1}{2} e^{-x^2} \Big|_0^2 \\ &= \frac{1}{2}\left(1 - e^{-4}\right) \end{aligned} \tag{24}$$