Calculus 3 - Greens Theorem

Last class we ended with the problem of trying to evaluate

$$\oint_C 2y\,dx + x\,dy \tag{1}$$

where C is along circle $x^2 + y^2 = 4$ in the CCW direction. We said the vector field is not conservative since

$$P = 2y, \quad Q = x \ \text{ and } \ Q_x = 1 \neq P_y = 2. \tag{2}$$

However, there is a nice theorem which relates the line integral over a vector field for closed curve to the region of the closed curve itself.

Green's Theorem

Let R be be simply connected region with a piecewise smooth boundary C, oriented counterclockwise. Let P and Q have continuous first partial derivatives in an open region containing R, then

$$\int_C P\,dx + Q\,dy = \iint_R \left(Q_x - P_y\right) dA \tag{3}$$

Example 1. Evaluate

$$\oint_C 2y\,dx + x\,dy \tag{4}$$

where C is along circle $x^2 + y^2 = 4$ in the CCW direction.

Soln.

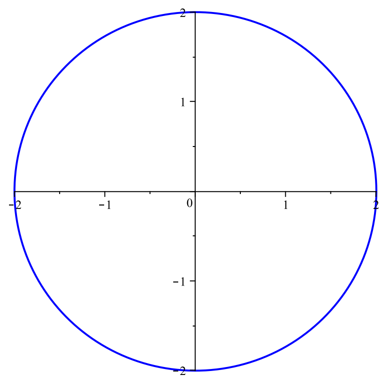
Since we saw that

$$Q_x = 1, \quad P_y = 2, \tag{5}$$

then

$$\oint_C 2y\,dx + x\,dy = \iint_R (1-2)dA = -\iint_R dA \tag{6}$$

Since the integrand is equal to 1, then the double integral is just he area of



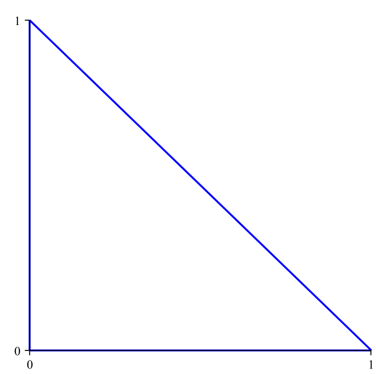
the region which is 4π so

$$\oint_C 2y\,dx + x\,dy = -4\pi \tag{7}$$

Example 2. Verify Green's theorem for

$$\oint_C x^4\,dx + xy\,dy \tag{8}$$

where R is the region bound by $y = 0, x = 0$, and $y = 1 - x$.



Soln.

We first do the line integral part. Here there are three curve so we do each one separately.

$C_1: \quad y = 0$:

Since $y = 0$, then $dy = 0$ and our line integral becomes

$$\int_0^1 x^4 \, dx = \frac{1}{5}x^5\Big|_0^1 = \frac{1}{5}. \tag{9}$$

$C_2: \quad y = 1 - x$:

Since $y = 1 - x$, then $dy = -dx$ and our line integral becomes

$$\int_1^0 x^4 \, dx - x(1-x) \, dx = \left(\frac{1}{5}x^5 - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)\Big|_1^0 = -\frac{1}{30}. \tag{10}$$

$C_3: \quad x = 0$: Along $x = 0$, then $dx = 0$ so the line integral is zero. Thus,

$$\oint_C x^4 \, dx + xy \, dy = \frac{1}{5} - \frac{1}{30} = \frac{1}{6}. \tag{11}$$

For the second part, we identify that $P = x^4$ and $Q = xy$ so

$$Q_x - P_y = y \tag{12}$$

so

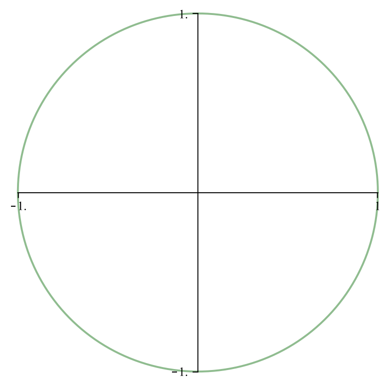
$$\begin{aligned} \int_0^1 \int_0^{1-x} y dy dx &= \int_0^1 \frac{1}{2}y^2\Big|_0^{1-x} dx = \int_0^1 \frac{1}{2}(1-x)^2 \, dx \\ &= -\frac{1}{6}(1-x)^3\Big|_0^1 = \frac{1}{6}. \end{aligned} \tag{13}$$

the same.

Example 3. Verify Green's theorem for

$$\oint_C y^3\,dx - x^3\,dy \tag{14}$$

where R is the region bound by the circle $x^2 + y^2 = 1$



Soln.

We first do the line integral part. Here we parameterize the circle with

$$x = \cos t, \quad y = \sin t, \tag{15}$$

so

$$dx = -\sin t\,dt, \quad dy = \cos t\,dt, \tag{16}$$

and the line integral becomes

$$\begin{aligned}\oint_C y^3\,dx - x^3\,dy &= \int_0^{2\pi} \sin^3 t \cdot (-\sin t dt) - \cos^3 t \cdot \cos t dt \\ &= -\int_0^{2\pi} \frac{3+\cos 4t}{4} dt = -\left(\frac{3}{4}t + \frac{1}{16}\sin 4t\right)\Big|_0^{2\pi} = -\frac{3}{2}\pi\end{aligned} \tag{17}$$

For the second part, we identify that $P = y^3$ and $Q = -x^3$ so

$$Q_x - P_y = -3x^2 - 3y^2 \tag{18}$$

so

$$-3 \iint\limits_R \left(x^2 + y^2\right) dA \tag{19}$$

Since the region is a circle, we switch to polar so

$$\begin{aligned} -3 \iint\limits_R \left(x^2 + y^2\right) dA &= -3 \int_0^{2\pi} \int_0^1 r^2 \cdot r\, dr\, d\theta \\ &= -3 \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^1 d\theta \\ &= -\frac{3}{4} \int_0^{2\pi} d\theta = -\frac{3}{4}\theta \Big|_0^{2\pi} = -\frac{3}{2}\pi \end{aligned} \tag{20}$$

the same.

Area of Plane Regions

We can also use Green's theorem to find the area of a region in the xy plane. Suppose that

$$Q_x - P_y = 1. \tag{21}$$

Then Green's theorem says

$$\int\limits_C P\, dx + Q\, dy = \iint\limits_R \left(Q_x - P_y\right) dA = \iint\limits_R 1 dA = A. \tag{22}$$

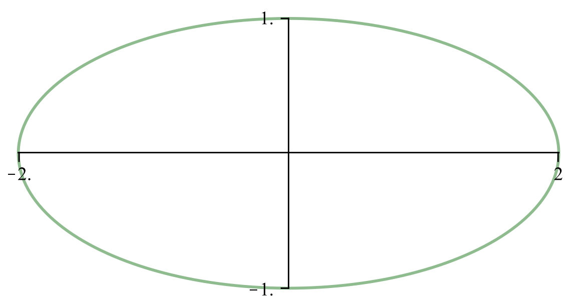
So as long as we choose P and Q so that it satisfies (21) then the line integral will give the area of the region. Here are some possibilities

$$\int_C x\,dy, \quad \int_C -y\,dx, \quad \int_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy \tag{23}$$

Example 4. Use Green's theorem to find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{24}$$

Soln.



Here we use

$$\int_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy \tag{25}$$

We parameterize the ellipse by

$$x = a\cos t, \quad y = b\sin t, \tag{26}$$

so

$$dx = -a\sin t\,dt, \quad dy = b\cos t\,dt \tag{27}$$

So (28) becomes

$$\begin{aligned}
&\int_0^{2\pi} -\frac{1}{2}\left(b\sin t\right)\left(-a\sin t\,dt\right)+\frac{1}{2}\left(a\cos t\right)\left(b\cos t\,dt\right) \\
&\qquad=\frac{ab}{2}\int_0^{2\pi}\left(\sin^2 t+\cos^2 t\right)dt \\
&\qquad=\frac{ab}{2}\int_0^{2\pi} dt=\pi ab
\end{aligned} \tag{28}$$

It $a=b=r$ then we get the area of a circle πr^2.