

Calculus 3 - Vector Functions

In Calculus 1, one application of the derivative is calculating velocity and acceleration. If $s = s(t)$ is position then we found that velocity is

$$v = \frac{ds}{dt} \quad (1)$$

and acceleration is

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \quad (2)$$

For example, with a falling body where $s = 16t^2$ then

$$v = \frac{ds}{dt} = 32t \quad (3)$$

and

$$a = \frac{dv}{dt} = 32 \quad (4)$$

With the introductions of vector functions, we define the velocity vector as

$$\vec{v} = \frac{d\vec{r}(t)}{dt} \quad (5)$$

and acceleration vector as

$$\vec{a} = \frac{d\vec{v}(t)}{dt} \quad (6)$$

Example 1 If $\vec{r} = \langle t, \frac{1}{2}t^2 \rangle$ then $\vec{r}' = \langle 1, t \rangle$ and so $\vec{v}' = \langle 1, t \rangle$. We also calculate $\vec{a} = \langle 0, 1 \rangle$

Example 2 If $\vec{r} = \langle \cos t, \sin t, t \rangle$ then $\vec{r}' = \langle -\sin t, \cos t, 1 \rangle$ so $\vec{v}' = \langle -\sin t, \cos t, 1 \rangle$. We further calculate $\vec{a} = \langle -\cos t, -\sin t, 0 \rangle$.

Yesterday we defined the unit Tangent and unit Normal vectors and were given by

$$\vec{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}, \quad \vec{N} = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}. \quad (7)$$

For example 1, they were calculated to be

$$\vec{T} = \frac{\langle 1, t \rangle}{\sqrt{1+t^2}}, \quad \vec{N} = \frac{\langle -t, 1 \rangle}{\sqrt{1+t^2}}, \quad (8)$$

They are shown in figure 1 (at $t = 1$) as well as the acceleration vector \vec{a} .

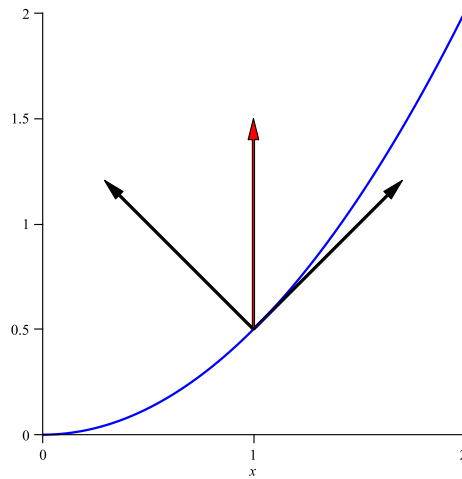


Figure 1: The vectors \vec{T}, \vec{N} and \vec{a}

It appears that's there a connection between the three. We, in fact, there is and is as follows:

$$\vec{a} = a_T \vec{T} + a_N \vec{N},$$

where

$$a_T = \frac{d\|\vec{r}'\|}{dt}, \quad a_N = \|\vec{r}'\| \|\vec{T}'\| \quad (9)$$

Proof

Since

$$\vec{T} = \frac{\vec{r}'}{\|\vec{r}'\|} \quad (10)$$

then

$$\vec{r}' = \|\vec{r}'\| \vec{T} \quad (11)$$

or

$$\vec{v} = \|\vec{r}'\| \vec{T} \quad (12)$$

Differentiating this with respect to t gives

$$\vec{v}' = \|\vec{r}'\|' \vec{T} + \|\vec{r}'\| \vec{T}' \quad (13)$$

Since

$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|} \quad (14)$$

then (13) becomes

$$\vec{a} = \|\vec{r}'\|' \vec{T} + \|\vec{r}'\| \|\vec{T}'\| \vec{N} \quad (15)$$

which is (9).

Example 1

$$\begin{aligned} \vec{r} &= \left\langle t, \frac{1}{2}t^2 \right\rangle \\ \vec{r}' &= \langle 1, t \rangle \\ \|\vec{r}'\| &= \sqrt{t^2 + 1}. \end{aligned}$$

so

$$\vec{T} = \frac{\vec{r}'}{\|\vec{r}'\|} = \left\langle \frac{1}{\sqrt{t^2 + 1}}, \frac{t}{\sqrt{t^2 + 1}} \right\rangle$$

Further

$$\begin{aligned}\vec{T}' &= \left\langle \frac{-t}{(t^2 + 1)^{3/2}}, \frac{1}{(t^2 + 1)^{3/2}} \right\rangle \\ \|\vec{T}'\| &= \frac{1}{t^2 + 1}.\end{aligned}$$

so

$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|} = \left\langle \frac{-t}{\sqrt{t^2 + 1}}, \frac{1}{\sqrt{t^2 + 1}} \right\rangle$$

The velocity and acceleration are given by

$$\begin{aligned}\vec{v} &= \vec{r}' = \langle 1, t \rangle \\ \vec{a} &= \vec{v}' = \langle 0, 1 \rangle.\end{aligned}$$

$$a_T = \|\vec{r}'\|' = \frac{t}{\sqrt{t^2 + 1}}, \quad a_N = \|\vec{r}'\| \|\vec{T}'\| = \frac{1}{\sqrt{t^2 + 1}}.$$

so

$$\begin{aligned}a_T \vec{T} + a_N \vec{N} &= \frac{t}{\sqrt{t^2 + 1}} \left\langle \frac{1}{\sqrt{t^2 + 1}}, \frac{t}{\sqrt{t^2 + 1}} \right\rangle + \frac{1}{\sqrt{t^2 + 1}} \left\langle \frac{-t}{\sqrt{t^2 + 1}}, \frac{1}{\sqrt{t^2 + 1}} \right\rangle \\ &= \langle 0, 1 \rangle \\ &= \vec{a}\end{aligned}$$

Example 2

$$\vec{r} = \langle \cos t, \sin t, t \rangle \quad \text{so} \quad \vec{r}' = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{r}'\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}.$$

so

$$\vec{T} = \frac{\vec{r}'}{\|\vec{r}'\|} = \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

Further

$$\begin{aligned} \vec{T}' &= \left\langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, 0 \right\rangle \\ \|\vec{T}'\| &= \sqrt{\frac{\sin^2 t}{2} + \frac{\cos^2 t}{2}} = \frac{1}{\sqrt{2}}. \end{aligned}$$

so

$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|} = \langle -\cos t, -\sin t, 0 \rangle$$

The velocity and acceleration are given by

$$\begin{aligned} \vec{v} &= \vec{r}' = \langle -\sin t, \cos t, 1 \rangle \\ \vec{a} &= \vec{r}'' = \langle -\cos t, -\sin t, 0 \rangle. \end{aligned}$$

So

$$a_T = \|\vec{r}'\|' = 0, \quad a_N = \|\vec{r}'\| \|\vec{T}'\| = \sqrt{2} \frac{1}{\sqrt{2}} = 1.$$

so

$$a_T \vec{T} + a_N \vec{N} = 0 \vec{T} + 1 \vec{N} = \vec{a}.$$