ARML Problem Set 1: Solutions
Due date: Monday, July 20, 2015

1. Compute the sum and product of the solutions to the equation

$$5 - |x| = \frac{6}{|x|}$$

Answer. Sum = 0, Product = 36
Solution. If x is a solution to the equation then $-x$ is also a root. Therefore, the sum of the roots is 0. The equation can be rearrange and factored as

$$(|x| - 2)(|x| - 3) = 0$$

Thus, the set of solutions is $x = \pm 2, \pm 3$. Therefore, the product of the roots is 36.

2. Let b be a positive integer such that $(24_b)^2 = 642_b$. Find b. Note: a_b is the number a in base b.

Answer. $b = 7$
Solution. By definition of bases,

$$\begin{aligned} (24_b)^2 &= (2b+4)^2 = 4b^2 + 16b + 16 \\ 642_b &= 6b^2 + 4b + 2 \end{aligned}$$

Therefore, equating them and solving for b gives -1 and 7. Since b is a positive integer, b must be 7.

3. A circle is inscribed in a square. There exist a point on the circle such that the distance from this point to the two nearest sides of the square is 1 and 2. Find the area of the square.

Answer. 100
Solution. Let the point in question be P, the center of the circle be O, and the radius of the circle be r. Next, form a right triangle with side length $r-1, r-2, r$. (Draw a diagram and it will be obvious). Therefore, solving

$$(r-1)^2 + (r-2)^2 = r^2$$

for r gives $r = 1$ and $r = 5$. It's obvious that $r > 4$. Therefore, the area is $(2r)^2 = 100$.

4. Consider an equilateral triangular piece of paper, XYZ, with side length 12. Let P be a point on YZ such that $YP = 3$. When X is folded over to the point P, a crease is formed. Find the length of this crease.

Answer. $\frac{39\sqrt{39}}{35}$
Solution. Let A be the intersection of the crease with XY and B be the the intersection of the crease with XZ. Let $XA = a$ and $XB = b$ then $AY = 12 - a$, $BZ = 12 - b$, $AP = a$, and $BP = b$. Applying Cosine Law to AYP gives

$$a^2 = 3^2 + (12-a)^2 - 2(12-a)(3)\cos 60°$$

yields $a = \frac{117}{21}$. Similarly, apply Cosine Law to BZP yields $\frac{117}{15}$. Finally, applying Cosine Law to YXZ gives

$$AB^2 = a^2 + b^2 - 2ab\cos 60°$$

Therefore, $AB = \frac{39\sqrt{39}}{35}$.

5. Find the number of integers a such that $a^2 + 2015$ is a perfect square.

Answer. 8
Solution. First, let a and k be positive integers such that

$$a^2 + 2013 = k^2$$

then

$$(a-k)(a+k) = 2015 = (5)(13)(31)$$

Thus

$$a = \frac{1}{2}(m+n), k = \frac{1}{2}(m-n)$$

where $m > n$ are positive divisors of 2015 and $mn = 2015$. There are $2^3 = 8$ positive divisors of 2015. Thus, there are 4 possible pairs of (m, n). Hence, there are 4 positive integer values of a. Therefore, there are 8 integers values of a.

Remark. This problem becomes more complicated if it's $a^2 + l$ where l is even.

6. Let $f(x)$ be a polynomial of degree 50 with roots $-25, -24, \ldots, -1, 1, 2, \ldots, 25$. Suppose that $f(26) = 1$. Find $f(0)$.

Answer. $-\binom{51}{25}^{-1}$
Solution. The polynomial must have the form

$$\begin{aligned} f(x) &= \frac{(x+25)(x+24)\cdots(x+1)(x-1)\cdots(x-24)(x-25)}{(26+25)(25+25)\cdots(26+1)(26-1)\cdots(26-24)(26-25)} \\ &= \frac{26(x+25)(x+24)\cdots(x+1)(x-1)\cdots(x-24)(x-25)}{51!} \end{aligned}$$

Thus,

$$f(0) = -\frac{(26)(25!)(25!)}{51!} = -\binom{51}{25}^{-1}$$

7. Let $a_1, \ldots, a_k$ be a finite sequence of positive integers such that

$$\sum_{i=1}^{k} a_i = 2015$$

Denote n_i as the number of the i's in $a_1, \ldots, a_k$. Find the maximum possible value of

$$\sum_{i=2}^{2015} (i-1)n_i$$

Answer. 2014
Solution. The sum of all the numbers can also be expressed as

$$\sum_{i=1}^{2015} in_i = 2015$$

Therefore,

$$\sum_{i=2}^{2015} (i-1)n_i = \sum_{i=1}^{2015} in_i - \sum_{i=1}^{2015} n_i = 2015 - \sum_{i=1}^{2015} n_i$$

Since there is at least one number, then $\sum_{i=1}^{2015} n_i \geq 1$. Therefore, $\sum_{i=2}^{2015} (i-1)n_i \leq 2014$. To get this desired result, let the sequence be $\{2015\}$.

Remark. There are several ways to do this problem. This solution is most likely the simplest one.

8. Find all primes of the form
$$p^4 + q^4 + r^4 - 3$$
where p, q, r are also primes.

Answer. 719
Solution. Without loss of generality, assume $p \leq q \leq r$.
Case 1: $p = q = r = 2$ then $2^4 + 2^4 + 2^4 - 3 = 45$, which is not prime.
Case 2: $p = q = 2$ and $r > 2$ then $p^4 + q^4 + r^4 - 3$ is even.
Case 3: $p, q, r > 2$ then $p^4 + q^4 + r^4 - 3$ is even.
Case 4: $p = 2$ and $q, r > 2$ then the desired value is of the form
$$q^4 + r^4 + 13$$
Case 4a: If $q \neq 3$ then $q, r \equiv \pm 1 \mod 6$. Thus, $q^4 + r^4 + 13 \equiv 3 \mod 6$, which is not a prime.
Case 4b: $q = 3$ then the desired value is of the form
$$r^4 + 94$$
Case 4ba: $r = 3$ then $3^4 + 94 = 175$, which is not a prime.
Case 4bb: $r = 5$ then $5^4 + 94 = 719$, which is a prime.
Case 4bc: $r > 5$ then $r^4 \equiv 1 \mod 5$. Thus, $r^4 + 94 \equiv 0 \mod 5$, which is not a prime.
Combining all the cases, there is only one prime of the desired form and that is 719.

Remark. Well...that was a case bash...I may have used too many cases.

9. Find the number of odd coefficients in the expansion of
$$(x + 1)^{2015}$$

Answer. 1024
Solution. Claim: Let n be a non-negative integer then
$$(x + 1)^{2^n} \equiv x^{2^n} + 1 \mod 2$$
Proof. This is by induction.
Base: $n = 0$ then $x + 1 \equiv x + 1 \mod 2$
Induction step: Assume the result is true for an arbitrary non-negative integer n then
$$(x+1)^{2^{n+1}} \equiv \left((x+1)^{2^n}\right)^2 \equiv \left(x^{2^n} + 1\right)^2 \equiv x^{2^{n+1}} + 2x^{2^n} + 1 \equiv x^{2^{n+1}} + 1 \mod 2$$
Therefore, by induction, the claim is true.

Using the above claim,
$$(x+1)^{2015} \equiv \frac{\prod_{n=0}^{10} (x+1)^{2^n}}{(x+1)^{2^5}} \equiv \frac{\prod_{n=0}^{10} (x^{2^n} + 1)}{x^{2^5} + 1} \mod 2$$
When the right hand side is expanded, there are $2^{10} = 1024$ terms. Therefore, there are 1024 odd coefficients.

Remark. Alternatively, you can use Lucas's Theorem...

10. Let ABC be a triangle such that $\angle ABC = 33°$ and $\angle ACB = 47°$. Let M be the midpoint of BC. A circle is tangent to BC at B and is also tangent to line AM. Let this circle intersect AB at B and P. Similarly, a circle is tangent to BC at C and is also tangent to line AM. Let this circle intersect AC at C and Q. Find $\angle APQ$.

Answer. 47°
Solution. Suppose the two circles are tangent to AM at X and Y then

$$BM = MX = MC = MY$$

Thus, $AX = AY$. Next, by power of a point,

$$(AP)(AB) = (AX)^2 = (AY)^2 = (AQ)(AC)$$

Thus,

$$\frac{AP}{AQ} = \frac{AC}{AB}$$

Hence, APQ is similar to ABC. Therefore, $\angle APQ = 47°$.

Remark. Depending on how you draw your diagram, there is a possibility where X is not strictly between A and M, but the answer and solution remain the same.