Math 1496 – Calc 1

Inverse Trig Functions

In our derivative section we found

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

so we define

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$$

we typically don't use $-\cos^{-1}x$

similarly

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \quad \text{so} \quad \int \frac{dx}{1+x^2} = \tan^{-1}x + c$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \quad \text{so} \quad \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + c$$

Ex 1 $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$?

Kinda Looks like $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

but there's a 4 in our problem?

I'm going to try a sub. to make it look like

let $x = au$

so $4-x^2 = 4-a^2u^2$ can I choose a so

that $4-a^2u^2 = 4-4u^2$?

why - so when I factor $4-4u^2 = 4(1-u^2)$

Well, if $a=2$

so $x = 2u$ $dx = 2du$

$$\int \frac{2du}{\sqrt{4-4u^2}} = \int \frac{2du}{\sqrt{4(1-u^2)}} = \int \frac{\cancel{2}du}{\cancel{2}\sqrt{1-u^2}}$$

$$= \sin^{-1} u + C = \sin^{-1}\left(\frac{x}{2}\right) + C$$

<u>ex 2</u> $\displaystyle\int \frac{dx}{9+16x^2}$ looks like $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1}x + C$

Again let $x = au$ so $dx = a\,du$

$$\int \frac{a\,du}{9+16a^2u^2}$$ ← I'd like the bottom to look like $1+u^2$

if I pick (Good!)

$16a^2 = 9$ so $9+16a^2u^2 = 9+9u^2 = 9(1+u^2)$

so $a = \frac{3}{4}$ so $x = \frac{3}{4}u$

my $\int$ $\displaystyle\int \frac{\frac{3}{4}du}{9+16\left(\frac{9}{16}\right)u^2} = \int \frac{\frac{3}{4}du}{9(1+u^2)} = \frac{1}{3\cdot 4}\tan^{-1}u + C$

so $= \frac{1}{12}\tan^{-1}\left(\frac{4}{3}u\right) + C$

Ex 3 $\displaystyle\int \frac{dx}{x\sqrt{x^4-1}}$

Kinda looks like $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C$

Comparing let $u = x^2$

Since $u^2 - 1 = x^4 - 1$

Now $du = 2x\,dx$ so $dx = \dfrac{du}{2x}$

$$\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{\frac{du}{2x}}{x\sqrt{u^2-1}} = \frac{1}{2}\int \frac{du}{x^2\sqrt{u^2-1}}$$

$$\frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1}|u| + C$$

so Ans $\frac{1}{2}\sec^{-1}(x^2) + C$

ex 4 $\displaystyle\int \frac{dx}{x^2+2x+2}$ the 2x is giving us a prob!

completing the square

$$x^2+2x+2 = (x^2+2x)+2$$
$$= (x^2+2x+1-1)+2$$
$$= (x^2+2x+1)-1+2$$
$$= (x+1)^2+1$$

so $\displaystyle\int \frac{dx}{(x+1)^2+1}$ let $u = x+1$
$du = dx$

$$\int \frac{du}{u^2+1} = \tan^{-1}u + c$$
$$= \tan^{-1}(x+1) + c$$

A comparison

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① $\int \frac{dx}{\sqrt{1-x^2}}$

② $\int \frac{x\,dx}{\sqrt{1-x^2}}$

③ $\int \frac{dx}{x\sqrt{1-x^2}}$

(1) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

(2) $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{}$

let $u = 1-x^2$

$du = -2x\,dx$

$\int \frac{-\frac{1}{2}du}{u^{1/2}} = -\sqrt{u}$

(3) we need more techniques (Calc 2)

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$$\int_{\sqrt{3}}^{3} \frac{dx}{x\sqrt{4x^2-9}}$$

let $x = \frac{3}{2}u$ so $4x^2-9 = 4\cdot\frac{9}{4}u^2-9 = 9(u^2-1)$

$dx = \frac{3}{2}\,du$

$$\int \frac{\cancel{\frac{3}{2}}\,du}{\cancel{\frac{3}{2}}u\sqrt{9(u^2-1)}} = \frac{1}{3}\int \frac{du}{u\sqrt{u^2-1}}$$

Now the limits $u = \frac{2}{3}x$ $\quad x = \sqrt{3} \quad u = \frac{2}{3}\sqrt{3} = \frac{2}{\sqrt{3}}$

$x = 3 \quad u = \frac{2}{3}\cdot 3 = 2$

$$\int_{\frac{2}{\sqrt{3}}}^{2} \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}u\Big|_{2/\sqrt{3}}^{2} = \sec^{-1}2 - \sec^{-1}\frac{2}{\sqrt{3}}$$

$$= \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}$$

$\sec^{-1}2 = \theta \qquad \sec^{-1}\frac{2}{\sqrt{3}} = \theta$

$\sec\theta = 2 \qquad \sec\theta = \frac{2}{\sqrt{3}}$

$\cos\theta = \frac{1}{2} \qquad \cos\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3} \qquad \theta = \frac{\pi}{6}$

(a little harder than some of the other probs)