Math 1496 - Calc I

L'Hôpital's Rule - 2

Here we consider several indeterminate forms we consider by example

"$0 \cdot \infty$"

Ex $\lim_{x \to 0^+} x \ln x =$ "$0 \cdot -\infty$"

Here we will move one term

$$\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \text{"}\frac{-\infty}{\infty}\text{"} \text{ now LH}$$

$$\lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x \to 0^+} \frac{1}{x} \cdot x^2$$

$$= -\lim_{x \to 0^+} x = 0$$

so $\lim_{x \to 0^+} x \ln x = 0$

"$\infty - \infty$"

Ex $\lim\limits_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$ simplify

$$\lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \text{"}\frac{0}{0}\text{"} \quad \text{now LH}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} = \text{"}\frac{0}{0}\text{"} \quad \text{LH again}$$

$$\lim_{x \to 0} \frac{-\sin x}{\cos x + \cos x - x \sin x}$$

$$\lim_{x \to 0} \frac{-\sin x}{2\cos x - x \sin x} = 0$$

So $\lim\limits_{x \to 0} \frac{1}{\sin x} - \frac{1}{x} = 0$

"0^0"

0^0 Ex $\lim_{x\to 0^+} x^x$

In this example & the rest, we need to move the power. Logs do this for us

Note $e^{\ln f} = f$

so $x^x = e^{\ln x^x} = e^{x \ln x}$

$$\lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x \ln x} = e^0 = 1$$

$\because \lim_{x\to 0^+} x \ln x = 0$ (1^{st} Ex)

so $\lim_{x\to 0^+} x^x = 1$

Note $0^0 \neq 1$ (not always)

$\lim_{x\to \infty} (e^{-x})^{\frac{1}{x}} =$ "0^0" but $= e^{-1}$

$$\text{“}\ 1^{\infty}\ \text{”}$$

$$\lim_{x\to 1} x^{\frac{1}{1-x}} \quad \text{“}\ 1^{\infty}\ \text{”}$$

$$x^{\frac{1}{1-x}} = e^{\ln x^{\frac{1}{1-x}}} = e^{\frac{1}{1-x}\ln x}$$

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = e^{\lim_{x\to 1}\frac{\ln x}{1-x}} = e^{-1}$$

Aside $\displaystyle \lim_{x\to 1} \frac{\ln x}{1-x} = \lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1$

“$\frac{0}{0}$” L.H.

So $\displaystyle \lim_{x\to 1} x^{\frac{1}{1-x}} = \frac{1}{e}$

"∞^0"

$$\lim_{x\to\infty}(1+x)^{\frac{1}{x}} = \text{"}\infty^0\text{"}$$

$$(1+x)^{\frac{1}{x}} = e^{\ln(1+x)^{\frac{1}{x}}} = e^{\frac{1}{x}\ln(1+x)}$$

Now $\displaystyle\lim_{x\to\infty}\frac{\ln(1+x)}{x} = \text{"}\frac{\infty}{\infty}\text{"}$ so L H

$$\lim_{x\to\infty}\frac{\frac{1}{1+x}}{1} = 0$$

$$\lim_{x\to\infty}(1+x)^{\frac{1}{x}} = e^0 = 1$$

So $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \text{"}1^{\infty}\text{"}$

$$\left(1+\frac{1}{x}\right)^x = e^{\ln\left(1+\frac{1}{x}\right)^x} = e^{x\ln\left(1+\frac{1}{x}\right)}$$

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \text{"}\infty\cdot 0\text{"}$$

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \text{"}\frac{0}{0}\text{"} \text{ so LH}$$

$$\lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(\cancel{-\frac{1}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = 1$$

so $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^{\lim\limits_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)} = e^1$