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Darkness for Dummies

## I. Introduction

In a pair of previous notes ("The Evolution of Darkness", and "Klein Geometry"), I have tried to explore fundamental issues associated with the MacDowell-Mansouri extension of the first-order Einstein-Cartan formalism for general relativity. The MacDowell-Mansouri version is a gauge theory which in many ways is more similar to QED, QCD, and the electroweak gauge theory than the textbook Einstein-Hilbert version of general relativity. Its action is quadratic in the field strength $F$, although its character is more like a topological $E \cdot B$ theory rather than the more familiar $E^2 - B^2$ structure of its particle-theory predecessors. In addition, spacetime is constructed from a vierbein, which in turn is found in the "coset" pieces of the MacDowell-Mansouri gauge potential $A$.

As discussed by Derek Wise, the mathematical underpinnings of this description originate with the Erlangen program for geometry created by Felix Klein in the 19th century. So there is a rich mathematical heritage underlying the formalism. Alas, I am not mathematical enough to exploit this feature directly. In the aforementioned notes, I have instead tried to infer the physics by study of simple examples. This note is devoted to an even simpler strategy, namely to concentrate on the situation in two spacetime dimensions, and to explore simple examples in even more detail than in the previous notes.

In the next section, I will summarize some basic results of the previous notes, but restricted to two spacetime dimensions. In Section III, I will introduce fermionic degrees of freedom into the story. Finally, in Section IV, I will re-introduce two compactified space dimensions in order to simulate in simple terms the six extra compactified space dimensions conjectured to underly the huge coefficient of $10^{120}$ in front of the MacDowell-Mansouri action. Section V is devoted to concluding comments.

## II. Review of the Basic Issues

The simplest system to study is deSitter space. In the Macdowell-Mansouri formalism, this spacetime is "pure gauge". No equations of motion or action principle is needed; only the requirement $F = 0$. We will choose the FRW form of the gauge potential:

$$A_\mu^{AB} = \begin{array}{c} 02 \\ 12 \\ 01 \end{array} \begin{pmatrix} H & 0 \\ 0 & Ha(t) \\ \hline 0 & k(t) \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} H \\ 0 \end{matrix}}\right\} \text{zweibein} \\ \} \text{Einstein-Cartan connection} \end{array}$$

The line element has the form

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \pm \frac{1}{H^2} A_\mu^{A2} A_\nu^{B2} \eta_{AB} \eta_{22}\, dx^\mu dx^\nu$$

$$\Rightarrow \eta_2 \left[ \eta_0 \eta_2\, dt^2 + \eta_1 \eta_2\, a^2(t)\, dx^2 \right] \Rightarrow dt^2 + \eta_1 a^2(t)\, dx^2$$

It is a matter of convention to demand that the coefficient of $dt^2$ be positive; this allows the insertion of the prefactor $\eta_2$ in the second line. The metric associated with the internal gauge group will in what follows be either Euclidean or Minkowski; our notation is as follows:

$$\eta_{AB} = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} = \eta_A \delta_{AB}$$

Again it is a matter of convention to choose $\eta_0 = +1$. Consequently the metric is Euclidean if the parameter $\eta_1 = +1$ and Minkowski if $\eta_1 = -1$.

For the record, there is another variant of the spacetime line element defined above, which seems to have simple, albeit enigmatic, properties. We will denote it by $d\Phi$ and call it the MacDowell-Mansouri line element:

$$d\Phi^2 = \frac{\eta_2}{H^2} A_\mu^{AB} A_\nu^{CD} \eta_{AC} \eta_{BD}\, dx^\mu dx^\nu = dt^2 + \eta_1 \left[ a^2(t) + \eta_2 \frac{k^2(t)}{H^2} \right] dx^2$$

Our definition of the field strength is as follows:

$$F_{\mu\nu}^{AB} = \partial_\mu A_\nu^{AB} - \partial_\nu A_\mu^{AB} + A_\mu^{AC} \eta_C A_\nu^{CB} - A_\nu^{AC} \eta_C A_\mu^{CB}$$

The conditions that F vanish lead to the following equations:

$$F_{tx} = \begin{array}{c} 02 \\ 12 \\ 01 \end{array} \begin{pmatrix} 0 \\ H\dot{a} + Hk\eta_0 \\ \dot{k} - H^2 a \eta_2 \end{pmatrix} = 0$$

These condense down to a single simple equation:

$$\ddot{a} = -H^2 a \eta_0 \eta_2 = -H^2 a \eta_2$$

This leads in turn to four cases to consider:

$$\eta_2 = +1: \quad \left[ \begin{array}{l} a = \sin Ht \\ k = -H \cos Ht \end{array} \right. \qquad \text{"Euclidean"}$$

$$\eta_2 = -1: \quad \left\{ \begin{array}{ll} \left[ \begin{array}{l} a = \sinh Ht \\ k = -H \cosh Ht \end{array} \right. & \text{"Closed"} \\ \left[ \begin{array}{l} a = \cosh Ht \\ k = -H \sinh Ht \end{array} \right. & \text{"Open"} \\ \left[ \begin{array}{l} a = e^{Ht} \\ k = -He^{Ht} \end{array} \right. & \text{"Flat"} \end{array} \right.$$

The Euclidean choice, $\eta_2 = +1$, is clean. The three cases for $\eta_2 = -1$ are in fact familiar from deSitter cartographies in four dimensional spacetime. They correspond to closed, open, and flat deSitter cosmologies, as indicated by the nomenclature adjacent to the equations.

With these results, we can also catalog the MacDowell-Mansouri line elements for the various cases. There are eight options, and in all cases the line elements are simple—especially so for the "flat" Minkowskian option of primary interest in this note:

$$d\Phi^2 = dt^2 \qquad \text{if} \quad \eta_2 = -1 \quad \text{and} \quad \eta_1 = \pm 1 \ \text{(flat)}$$

$$d\Phi^2 = dt^2 + dx^2 \left\{ \begin{array}{llll} \text{if} & \eta_2 = +1 & \text{and} & \eta_1 = +1 \ \text{(Euclidean)} \\ \text{or if} & \eta_2 = -1 & \text{and} & \eta_1 = +1 \ \text{(open)} \\ \text{or if} & \eta_2 = -1 & \text{and} & \eta_1 = -1 \ \text{(closed)} \end{array} \right.$$

$$d\Phi^2 = dt^2 - dx^2 \left\{ \begin{array}{llll} \text{if} & \eta_2 = +1 & \text{and} & \eta_1 = -1 \ \text{(Euclidean)} \\ \text{or if} & \eta_2 = -1 & \text{and} & \eta_1 = +1 \ \text{(closed)} \\ \text{or if} & \eta_2 = -1 & \text{and} & \eta_1 = -1 \ \text{(open)} \end{array} \right.$$

Certainly there is some message in the above pattern. We will however not pause to contemplate this curious simplicity further. But we may return to this subject later on. In the material to follow, we specialize to the "flat" choice $\eta_1 = \eta_2 = -1$.

The central focus in previous notes resides in the concept of "darkness". It is defined as the number of purported topological structures inhabiting a given volume of space. For deSitter space in 3 + 1 dimensions, the darkness density scales as $H M_{pl}^2 \cong \Lambda_z^3$. Here $\Lambda_z$ is the Zeldovich scale of about 20 MeV or $10^{-12}$cm. In this note featuring only one space dimension, we will assume that the darkness density is again of order 1 per $10^{-12}$cm, and that the Hubble scale H associated with the accelerated expansion of space to be of order $10^{28}$cm. (for $H^{-1}$).

In this reduced case, we define the darkness in terms of the Einstein-Cartan O(1,1) curvature R. The Gauss-Bonnet Lagrangian is

$$L = 2\pi \frac{dN}{dt} = C\int dx\, R^{01}_{tx} = C\int \dot{k}\, dx = C\int k(t)\, dt$$

$$N(t) = \frac{C}{2\pi}\int k\, dx = \frac{CH}{2\pi}\int a(t)\, dx = \frac{CH\ell(t)}{2\pi} \sim \Lambda_z\, \ell(t)$$

Again, we have assumed that the dimensionless coefficient C of the Gauss-Bonnet action is large, so that the darkness density comes out according to the criteria we specified:

$$n(t) = \frac{N(t)}{\ell(t)} \sim \Lambda_z \qquad\qquad C \sim \frac{\Lambda_z}{H} \sim 10^{40}$$

Note that, in the original 4-dimensional case, the coefficient C could have been written in the same form:

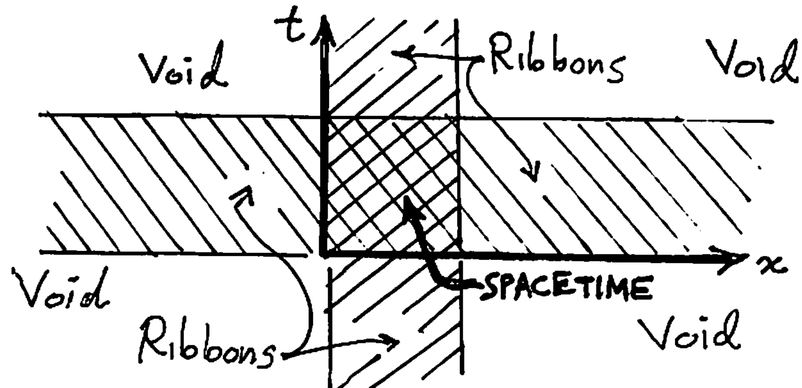
$$C \sim \frac{M_{pl}^2}{H^2} = \frac{M_{pl}^2 H}{H^3} \sim \left(\frac{\Lambda_z}{H}\right)^3$$

Choosing this form would excise all direct reference to the Planck scale from the MacDowell-Mansouri effective action.

A basic issue addressed in the previous note was the relation of spacetime, as defined via the line element, to the input substrate variables---in this case x and t . We found that we could, via gauge transformations, stop the FRW clock and separate pieces of spacetime from each other. Furthermore, we were led to identify each unit of darkness with a "fundamental volume" of space, scaled by the Zeldovich parameter. Each fundamental volume contained a "fundamental energy" of order H. In an expanding deSitter universe, this value of the energy is irreducible, since to measure it better would take a time long compared to the time the universe doubles in size. In this simpler version, the fundamental volume is replaced by a fundamental length or size, again of order the Zeldovich scale.

In the previous note, the disassembly of deSitter space led to isolated spacetime islands within the substrate connected to each other by "rods", which in turn were connected to each other by "plates". In a rod, the vierbein component along the axis of the rod vanished, while the other two vierbein space components remained nonvanishing. In a plate, only the vierbein component normal to the plate remained nonvanishing. In void regions outside spacetime islands, rods, and plates, all components of the vierbein vanished.

In our simplified case, we only have spacetime islands, "ribbons", and voids to deal with. In the FRW geometry, the disassembly of deSitter space in terms of ribbons and voids is shown below.




The gauge potential in the various regions is.

$$\text{Voids:} \quad A_\mu^{AB} = \begin{array}{c} \\ 02 \\ 12 \\ 01 \end{array} \begin{array}{c} \begin{array}{cc} t & x \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

$$\text{Spacetime:} \quad A_\mu^{AB} = \begin{pmatrix} 1 & 0 \\ 0 & e^t \\ 0 & -e^t \end{pmatrix}$$

$$\text{Horizontal ribbons:} \quad A_\mu^{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

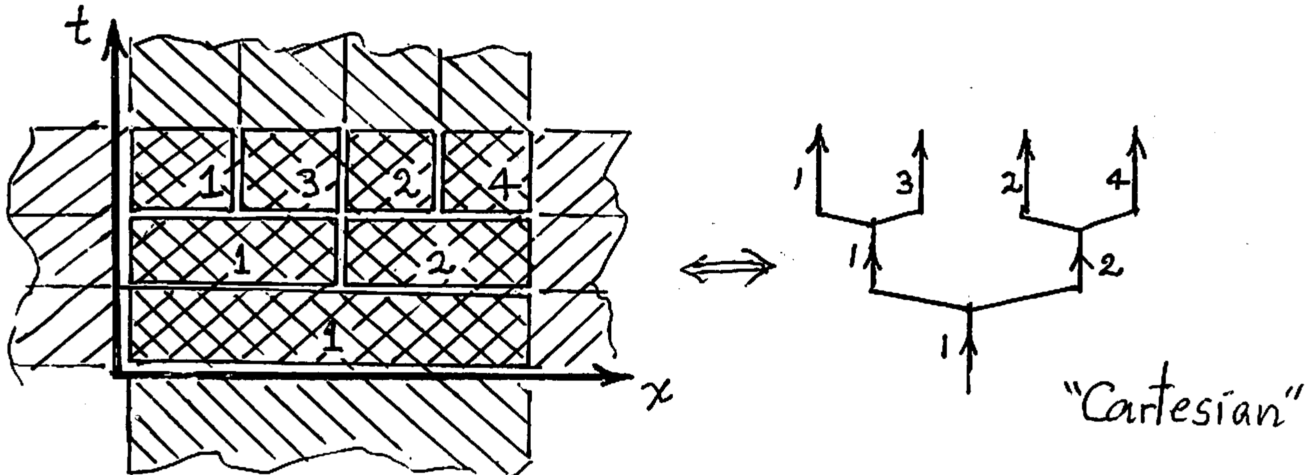
$$\text{Vertical ribbons:} \quad \begin{cases} A_\mu^{AB} = \begin{pmatrix} 0 & 0 \\ 0 & e^T \\ 0 & -e^T \end{pmatrix} \quad (\text{top}) \\[2em] A_\mu^{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \quad (\text{bottom}) \end{cases}$$

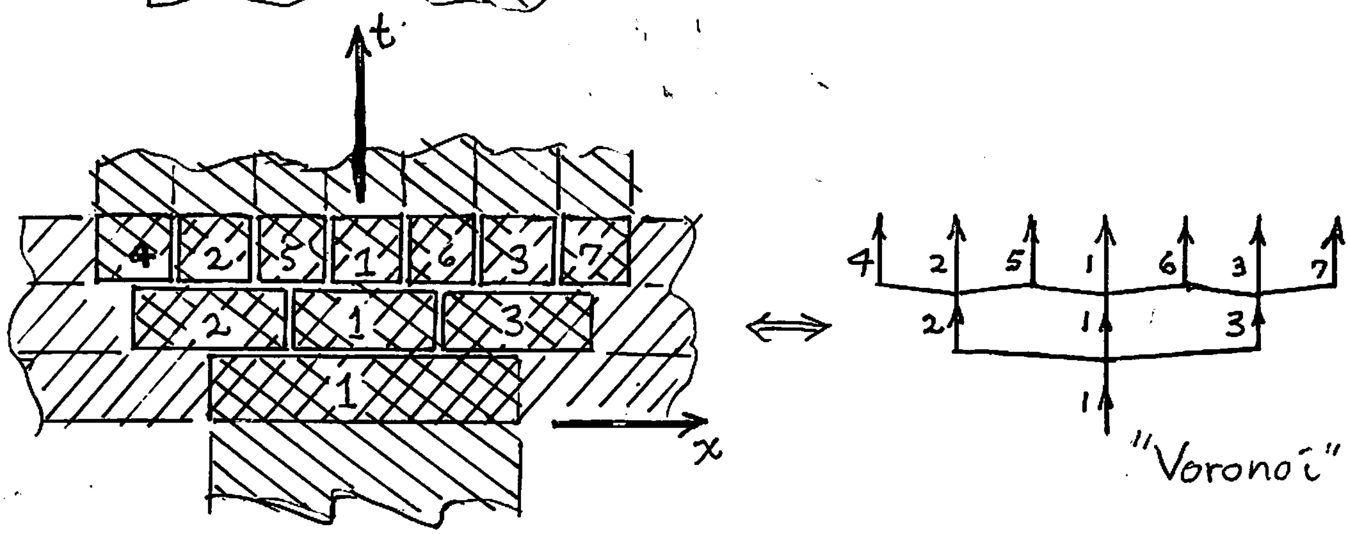
Note that

$$\int_{-\infty}^{\infty} A_0^{AB}(t)\,dt \qquad \begin{array}{c} 02 \\ 12 \\ 01 \end{array}\begin{pmatrix} T \\ 0 \\ 0 \end{pmatrix}$$

$$\int_{-\infty}^{\infty} A_x^{AB}(t)\,dt = \begin{cases} \begin{pmatrix} 0 \\ \mathrm{X}\, e^{T} \\ -\mathrm{X}\, e^{T} \end{pmatrix} & t \geq T \\[2ex] \begin{pmatrix} 0 \\ \mathrm{X}\, e^{t} \\ -\mathrm{X}\, e^{t} \end{pmatrix} & 0 \leq t \leq T \\[2ex] \begin{pmatrix} 0 \\ \mathrm{X} \\ -\mathrm{X} \end{pmatrix} & t \leq 0 \end{cases}$$

Finally, in the previous note we created a speculative description of inflation by exhibiting the growth of darkness with time in terms of a genealogical tree. Two versions were presented there. We do not reproduce the details here—only the tree structures for the two cases ("Cartesian" and "Voronoi") which were considered:



However, in what follows we will only consider the "Cartesian" genealogy. But we should recall that the motivation for the introduction of such an arcane description had to do with visualizing the inflation process and especially its "reheating" termination. The basic idea is that, at the onset of an inflationary episode, the fraction of substrate occupied by spacetime is very small. At "reheating", the fraction of substrate occupied by spacetime becomes of order unity. Consequently, inflationary expansion thereafter becomes impossible.

## III. Introduction of Fermions

The role of fermions in topology-rich field theory scenarios is typically very large. One sees this very clearly in QCD, where the topological currents of pure Yang-Mills theory mix with the axial vector currents of the quarks. It is therefore very natural to include them in the context of the MacDowell-Mansouri description of gravity.

When we contract the gauge potentials into gamma matrices, they can for deSitter space be written as follows:

$$A_\mu \Rightarrow \frac{\gamma_A \gamma_B}{4} A_\mu^{AB} \equiv A_\mu = U^{-1}(x,t)\,\partial_\mu U(x,t)$$

Factors of two are troublesome in what follows. If in doubt write things out! The field strength takes the form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

For the FRW description which we have used, the form of U(x,t) was worked out in previous notes. The gamma matrices for our case are 2 x 2 . The matrix U can be written as follows (We often set H = 1 in what follows.):

$$U = \begin{pmatrix} e^{t/2} & 0 \\ x e^{t/2} & e^{-t/2} \end{pmatrix} \qquad U^{-1} = \begin{pmatrix} e^{-t/2} & 0 \\ -x e^{t/2} & e^{t/2} \end{pmatrix}$$

With this form, we easily find

$$A_t = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad A_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} e^t$$

We connect with our original notation provided we define

$$\gamma_0\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma_1\gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \gamma_1\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The gamma matrices consistent with these choices are all pure imaginary:

$$\gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The information in the matrix U can also be expressed in terms of spinor degrees of freedom. Define

$$\chi_{up} = U^{-1}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e^{-t/2}\\ -x e^{t/2}\end{pmatrix} \qquad \chi_{down} = U^{-1}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\ e^{t/2}\end{pmatrix}$$

These are known as parallel (or Killing?) spinors, and satisfy the equation

$$(\partial_\mu + A_\mu)\chi_\alpha = 0$$

This implies that they also satisfy the massless Dirac equation with gauge-invariant substitution included. Therefore these parallel spinors are in a sense a kind of zero mode fermion. However, they are not orthogonal, do not normalize nicely, and blow up as FRW time goes to infinity. Much of this pathology is due to the fact that the matrix U is not unitary.

In any case, the properties of these spinors do not seem to match what can be expected from the usual O(3,1) description. In that case, the description begins by writing down a Lagrangian-- something we did not need to do for the parallel spinors. It is ( for the case of 3 + 1 dimensional spacetime!!)

$$\begin{aligned}\mathcal{L} &= \|e\|\, e^\mu_A\, \overline{\Psi}\gamma^A\left(\partial_\mu + \frac{\gamma_B\gamma_C}{4}\omega_\mu^{BC}\right)\psi \\ &= a^3\overline{\Psi}\left[\frac{1}{H}\gamma^0\frac{\partial}{\partial t} + \frac{1}{a}\gamma^i\left(\frac{\partial}{\partial x_i} + \frac{\gamma_0\gamma_i}{2}k\right)\right]\psi \\ &= a^3\overline{\Psi}\left[\gamma_0\frac{\partial}{\partial t} + \frac{\vec{\gamma}\cdot\vec{\nabla}}{a} - \frac{3}{2}\gamma_0\right]\psi\end{aligned}$$

The equation of motion which follows from this is (watch for deadly minus signs and factors of two!)

$$\left[\frac{\partial}{\partial t} + \frac{1}{a(t)}\gamma^0\vec{\gamma}\cdot\vec{\nabla} - \frac{3}{2}\right]\psi = 0$$

This result of course resides in the literature, e.g. in arXiv 1101.3164.

For deSitter space, there is an exact solution in terms of Hankel functions. But it is not terribly enlightening. For a zero-momentum spinor, the answer is easy:

$$\psi \sim e^{-3/2\, Ht}$$

This solution is normalizable. The probability of finding the particle within a comoving volume element is independent of time:

$$\int d^3x \; a^3(x) \, \psi^\dagger \psi = \text{constant}$$

Note that in one space dimension, the zero-momentum solution becomes

$$\psi \sim e^{-\frac{1}{2} Ht}$$

Again this solution is normalizable. (This is why we bothered to exhibit the O(3,1) generalization here.)

For nonzero comoving momenta, we can expect nontrivial complications, because in the distant past the Dirac particle has high physical momentum, and is almost unaffected by the dark energy. However as time passes the particle's momentum and energy is redshifted, and it eventually "crosses the horizon". In phase space its Wigner function no longer looks like that of a harmonic oscillator but evolves as a highly squeezed state. None of these features seem to be present for the parallel spinors, suggesting that we must think of them in very different terms from the spinors which describe "real particles". In addition, real spin-1/2 particles all have mass, and therefore have spheres of influence around them. The only possible exception among the known quarks and leptons is the first-generation neutrino.

I am left with a real conundrum—is there any point in introducing and worrying about the parallel spinors? At this point I simply do not know.