Calculus 3 - Line Integrals

Consider the following. Suppose we had a straight wire with a constant density, ρ and length l, what would be the mass of the wire? This is a basic physics problem were we have

$$m = \rho \times l. \tag{1}$$

Now suppose that the density of the wire changes with respect to its length, say $\rho = f(x)$, what would the mass now be? If we assume that over a very small interval, the density is approximately constant, then over this small length (we'll call this dx) the mass would be

$$dm = f(x)dx, \tag{2}$$

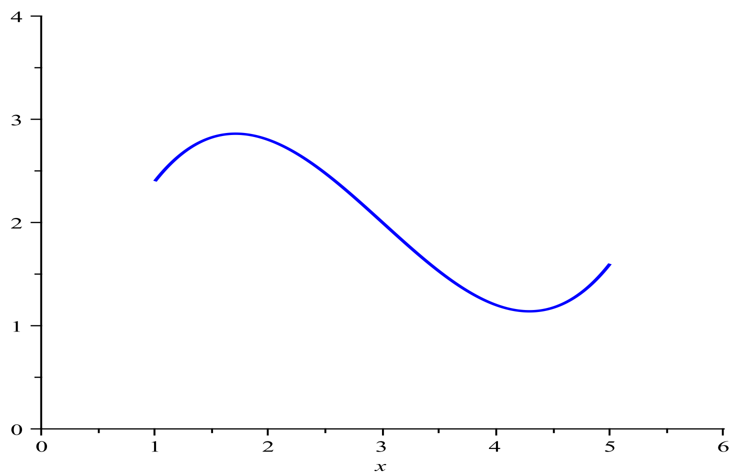
and adding all the small masses would give, in the limit

$$m = \int_a^b f(x)dx. \tag{3}$$

These are 1D problems. Suppose now the wire is two-dimensional and has as its the density $\rho = f(x, y)$ and bent in the shape of some curve C. What would be the mass of the wire now?

If we consider a small part of this curve, say ds, and assuming over this small part the density is constant, we would have the mass

$$dm = f(x, y)ds$$



and adding all the little masses up gives, in the limit

$$m = \int\limits_C f(x, y) ds. \tag{4}$$

This is called a *line integral*.

Recall from arc length, that

$$ds = \sqrt{1 + y'^2}\, dx \tag{5}$$

and so we have

$$\int\limits_C f(x, y) ds = \int_a^b f(x, y(x)) \sqrt{1 + y'^2}\, dx \tag{6}$$

Example 1. Evaluate

$$\int\limits_C \left(x^2 + y^2\right) ds \tag{7}$$

where C is the straight line from $(0, 0)$ to $(2, 4)$.

Soln.

We first need the curve. As we have a straight line, we easily obtain

$$y - 4 = \frac{4}{2}(x - 2) \quad \Rightarrow \quad y = 2x \tag{8}$$

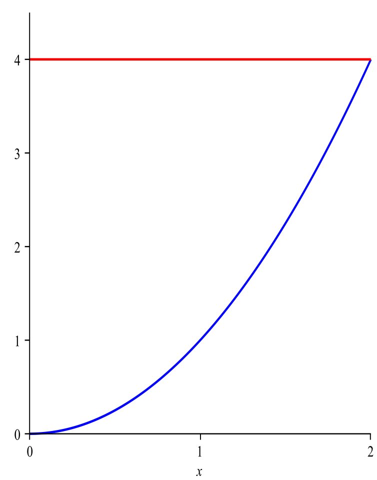
Next, since $y' = 2$, then $ds = \sqrt{1 + 2^2}dx = \sqrt{5}\,dx$. The limits of integration are $x = 0 \to 2$. Thus, (7) becomes

$$\int_0^2 \left(x^2 + (2x)^2\right)\sqrt{5}\,dx = \sqrt{5}\int_0^2 5x^2 dx = \frac{40\sqrt{5}}{3}. \tag{9}$$

Example 2. Evaluate

$$\int_C 2x ds \tag{10}$$

where C is the parabola $y = x^2$ from $(0,0)$ to $(2,4)$ followed by the straight line from $(2,4)$ to $(0,4)$.



Soln.

We now have two curves and so we'll need two integrals, one for each curve.

C_1: Here $y = x^2$ so $y' = 2x$ and we have

$$\int_0^0 2x\sqrt{1+4x^2}dx = \frac{1}{6}\left(1+4x^2\right)^{3/2}\Big|_0^1 = \frac{17\sqrt{17}-1}{6} \tag{11}$$

C_2: Here $y = 4$ so $y' = 0$ and $ds = 1dx$and we have

$$\int_2^0 2xdx = -\int_0^2 2xdx = -x^2\Big|_0^2 = -4 \tag{12}$$

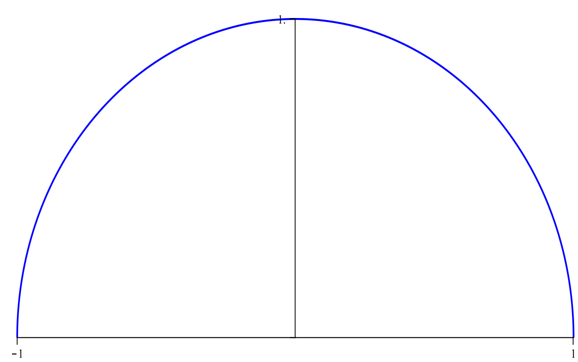
and so

$$\int_C 2xds = \frac{17\sqrt{17}-1}{6} - 4 = \frac{17\sqrt{17}-25}{6} \tag{13}$$

Example 3. Evaluate

$$\int_C (2+x^2y)ds \tag{14}$$

where C is the upper half circle $x^2+y^2 = 1$ from $(1,0)$ to $(-1,0)$.



Soln.

We certainly could solve for y giving $y = \sqrt{1-x^2}$ but things get a little complicated. However, we can parameterize the circle by

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le \pi, \tag{15}$$

so we need to modify the line integral formula (6). If x and y are given parametrically $x = x(t)$ and $y = y(t)$ then we have

$$\int_C f(x,y)ds = \int_{t_1}^{t_2} f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{16}$$

For example 3 we have

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t \tag{17}$$

and so

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{\sin^2 t + \cos^2 t} = 1 \tag{18}$$

and (16) becomes

$$\int_0^{\pi} \left(2 + \cos^2 t \sin t\right) dt = 2t - \frac{1}{3}\cos^3 t\Big|_0^{\pi} = 2\pi + \frac{2}{3}. \tag{19}$$

Line Integrals in Space

We now consider line integrals when

$$x = x(t), \quad y = y(t), \quad z = z(t). \tag{20}$$

The line integral in this case is

$$\int_c f(x,y,z)ds \tag{21}$$

In 2D

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{22}$$

in 3D

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{23}$$

and so (21) becomes

$$\begin{aligned} &\int_c f(x,y,z)ds \\ &= \int_{t_1}^{t_1} f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \end{aligned} \tag{24}$$

Example 4. Evaluate

$$\int_C xe^{yz} ds \tag{25}$$

where C is the line from $(0,0,0)$ to $(1,2,3)$.

Soln.

We first need the equation of the line. It follows the vector

$$\overrightarrow{PQ} = \langle 1,2,3 \rangle \tag{26}$$

The equation of the line is

$$x = t, \quad y = 2t, \quad z = 3t, \quad 0 \le t \le 1. \tag{27}$$

so

$$ds = \sqrt{1^2 + 2^2 + 3^2}\, dt = \sqrt{14}\, dt \tag{28}$$

$$\int_0^1 te^{6t^2}\sqrt{14}\, dt = \frac{\sqrt{14}}{12}\, e^{6t^2}\Big|_0^1 = \frac{\sqrt{14}}{12}\left(e^6 - 1\right) \tag{29}$$