VANDERBILT UNIVERSITY School of Engineering

Discrete Structures

CS 2212

(Fall 2020)

5 – Logic

Reminder and Recap …

Reminder: ZyBook Assig. 2A due Sep. 13 (11:59 PM)

HW 2 out. (Due on Sep 22.)

Recap:

English statements

Statements with nested quantifiers

Prove:

Premise ∧ Premise ∧ ⋯ ∧ Premise → Concl.

Rules of inference:

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array} \qquad \begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array} \qquad \begin{array}{l} p \\ \hline \therefore p \vee q \end{array} \qquad \begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array} \qquad \begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array} \qquad \begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array} \qquad \begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

Logic and Predicates: Proofs

Example: Prove that the argument with premises **$A \vee C \rightarrow D$**, **$\neg B$**, **$A \vee B$** and with the conclusion **D** is valid.

What we're really being asked to do is prove…

$$(A \vee C \rightarrow D) \wedge \neg B \wedge (A \vee B) \rightarrow D \text{ is true.}$$

Line	Statements	Why?
1	$A \vee C \rightarrow D$	Premise
2	$\neg B$	Premise
3	$A \vee B$	Premise
4	A	2, 3, Disjunctive Syll.
5	$A \vee C$	4, Addition
6	D	1, 5, Modus Ponens
7	QED	1-6

Logic and Predicates: Proofs

A Practice Problem:

Prove that the argument with

Premises : $(\neg p \land q)$, $(r \to p)$, $(s \to t)$, $(\neg r \to s)$, and
Conclusion: t

is a valid argument.

Proofs

Just a reminder.

$\equiv$	vs.	$\rightarrow$

Equivalence

P ≡ Q

Conditional

P → Q

So far, we have seen proofs in two contexts:

1. Proving that two statements are equivalent (**equivalence proofs**).
2. Proving that if a statement is true, then it implies some conclusion (**conditional proofs**).

Indirect Proofs*

Our goal is to prove: **A → B**

- So far, we have seen how to **use inference rules** and show that hypotheses on L.H.S imply the conclusion on the R.H.S.
- There is an another interesting way – **Indirect proofs**.
- First recall two facts:

1. A proposition cannot be true and false at the same time.

   $(A \wedge \neg A)$ = False (a contradiction).

2. If $(A \rightarrow B)$ then $(\neg B \rightarrow \neg A)$. Recall modus tollens.

   In words, if A is true, we know B is true. B is necessary for A. Consequently, if B is false, A must be false. Hence, $(\neg B \rightarrow \neg A)$.

(* Not in ZyBook)

Indirect Proofs - Approach

Our goal is to prove: **$A \rightarrow B$**

- May be it is difficult to "simplify" A and show A implies B.
- So, we use an **alternate approach (indirect proof)**.

We assume B is not true, that is $\neg$ B.

Then we prove using rules of inference that $\neg B \rightarrow \neg A$

(May be showing $\neg B \rightarrow \neg A$ is easier and straightforward as compared to showing $A \rightarrow B$.)

But we know that A is true as it is a given premise. However, in the above step we showed that A is false if I assume that B false.

Since A can't be true and false at the same time, my assumption that B is false is wrong.

Thus, B is true if A is true.

Hence $A \rightarrow B$

Indirect Proofs

Summary:

Prove: $A \rightarrow B$

1. Assume: $\neg B$
2. Show: $\neg B \rightarrow \neg A$
3. Observe: A is a premise, and $(A \wedge \neg A) = \text{False}$
4. Therefore: $\neg B$ is false
5. Hence: B is true, and $A \rightarrow B$

Indirect Proofs - Example

Prove: If $3n+2$ is odd, then n is odd

P: $3n+2$ is odd

Q: n is odd

We need to show: $P \rightarrow Q$

Lets try a **direct approach** first.

1. $3n+2$ is odd. — Premise
2. $3n+2 = 2k+1$ — By the definition of odd numbers
3. ???? — ???

There does not seem to be a direct way to conclude from here that n is odd. Lets try our new approach

Indirect Proofs - Example

Prove: If $3n+2$ is odd, then n is odd

P: $3n+2$ is odd

Q: n is odd

Show: $P \rightarrow Q$

1.	P	Premise
2.	$\neg$ Q (n is even).	Assumption
3.	$n = 2k$	By the definition of even numbers
4.	$3n+2 = 3(2k) + 2$	Replacing n in $(3n+2)$
5.	$2(3k+1)$	Simplifying line 3
6.	$2(3k+1)$ is even	By the definition of even numbers
7.	$\neg$ P	From line 5
8.	$P \wedge \neg P$ = False	1,7, Contradiction
9.	$P \rightarrow Q$	QED.

Indirect Proofs - Example

Lets look at another example of indirect proofs.

Prove: $(A \lor C \to D) \land \neg B \land (A \lor B) \to D$

Previously, we proved it using a direct approach. Now, we use an indirect approach.

Prove: **$(A \vee C \rightarrow D) \wedge \neg B \wedge (A \vee B) \rightarrow D$**

Line	Statements	Why?
1	$A \vee C \rightarrow D$	Premise
2	$\neg B$	Premise
3	$A \vee B$	Premise
4	$\neg D$	Assumption
5	$\neg (A \vee C)$	1, 4, Modus tollens
6	$\neg A \wedge \neg C$	5, DeMorgan's Law
7	$\neg A$	6, Conjunction
8	$\neg A \wedge \neg B$	2,7
9	$\neg (A \vee B)$	8, DeMorgans Law
10	$\neg(A \vee C \rightarrow D) \vee \neg(\neg B) \vee \neg (A \vee B)$	9, Disjunction
11	$\neg((A \vee C \rightarrow D) \wedge (\neg B) \wedge (A \vee B))$	10, DeMorgans Law
12	$(A \vee C \rightarrow D) \wedge \neg B \wedge (A \vee B)$	1,2,3 (Hypotheses)
13	False	11,12, Contradiction
14	D	13

Proofs with Quantifiers

How can we prove statements that involve quantifiers?

Good news is that whatever we have learned so far regarding conditional statement proofs, remains valid here. We just need few additional tools to make it work nicely.

Proofs with Quantifiers

How can we prove statements that involve quantifiers?

Good news is that whatever we have learned so far regarding conditional statement proofs, remains valid here. We just need few additional tools to make it work nicely.

Example: Every CS2212 persons own a laptop. Waseem is a CS2212 person. Therefore, Waseem owns a laptop.

x : person

$A(x)$: x is a CS2212 person.

$P(x)$: x owns a laptop

$$\forall x\,(A(x) \rightarrow P(x)) \wedge A(\text{Waseem}) \rightarrow P(\text{Waseem})$$

Proofs with Quantifiers

$$\forall x\,(A(x) \rightarrow P(x)) \wedge A(\text{Waseem}) \rightarrow P(\text{Waseem})$$

Given | Conclusion

1	$\forall x\,(A(x) \rightarrow P(x))$	Premise
2	A(waseem)	Premise
3	A(waseem) → P(waseem)	Since 1 is true for every x, so using a particular value of x = Waseem, it should be true.
4	P(waseem)	2,3, Modus ponens
5	QED	1 - 3

Proofs with Quantifiers

So, the trick lies in figuring out when and how to

1. **eliminate a quantifier**
2. **add a quantifier**

- An **arbitrary** element of a domain is an element that shares all of the characteristics of every other element in a domain
- A **particular** element of a domain is an element that possesses some characteristic not necessarily shared by all other elements.

Lets see some **rules of inference** involving quantifiers

Rules of Inference with Quantifiers

$\exists x \, P(x)$

∴ (c is a particular element) $\wedge$ P(c)

Existential instantiation

(Eliminating a quantifier)

Note that **c** cannot be an arbitrary value here

c is an element

P(c)

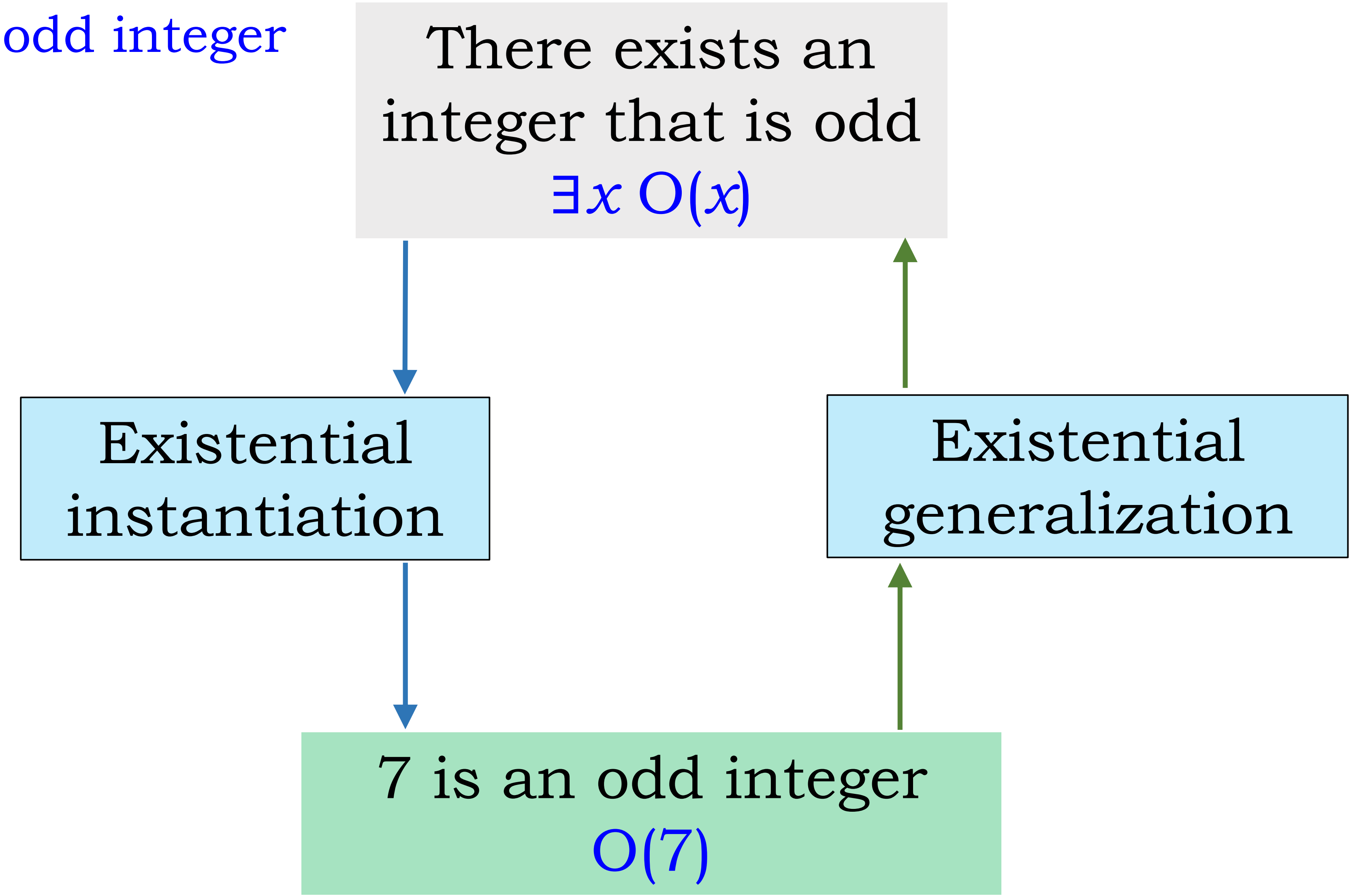
∴ $\exists x \, P(x)$

Existential generalization

(Adding a quantifier)

Rules of Inference with Quantifiers

$O(x)$ = x is an odd integer



Rules of Inference with Quantifiers

c is an element in the domain
$\forall x\, P(x)$

$\therefore P(c)$

Universal instantiation
(Eliminating a quantifier)

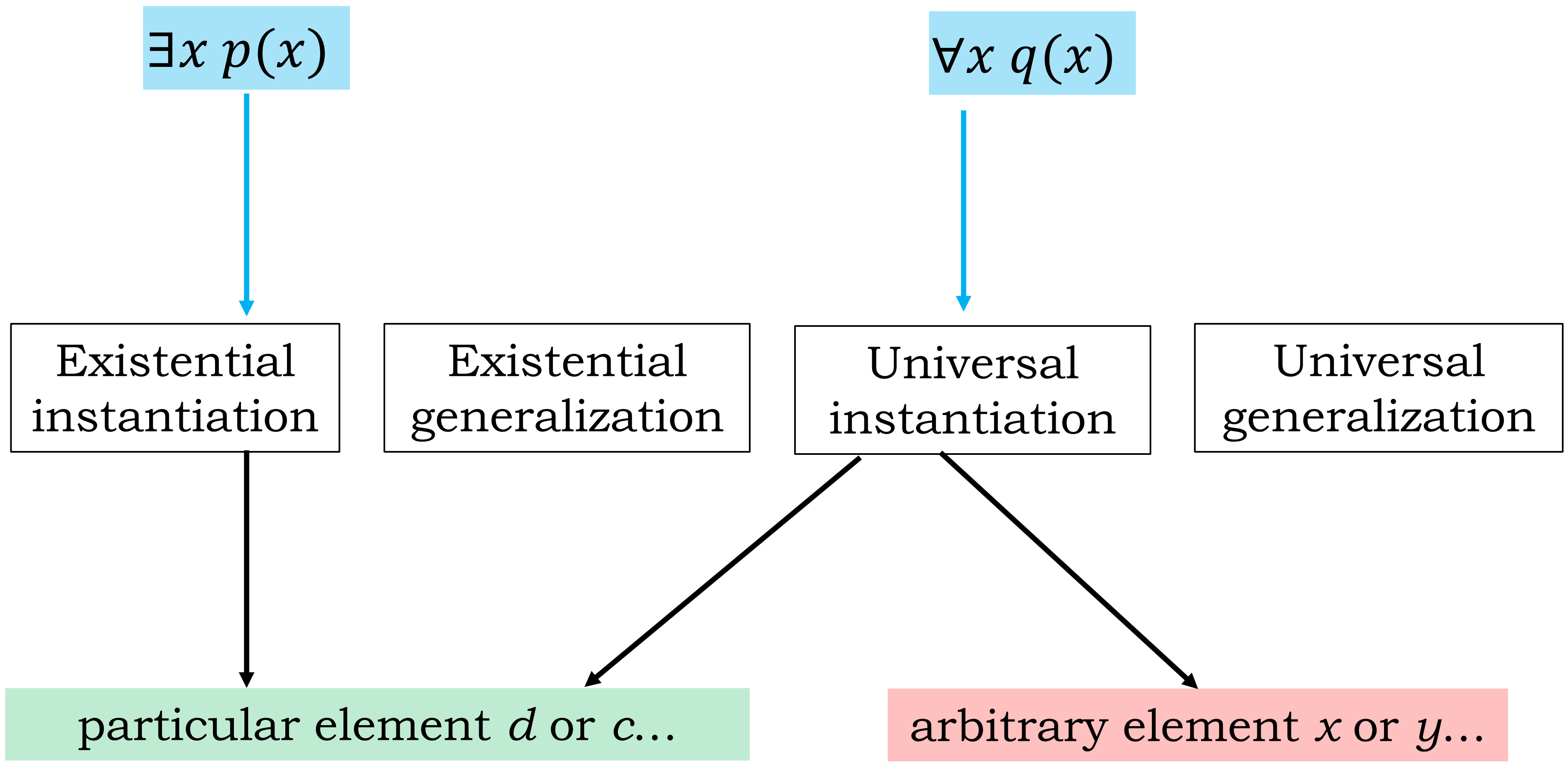
c is an arbitrary element
$P(c)$

$\therefore \forall x\, P(x)$

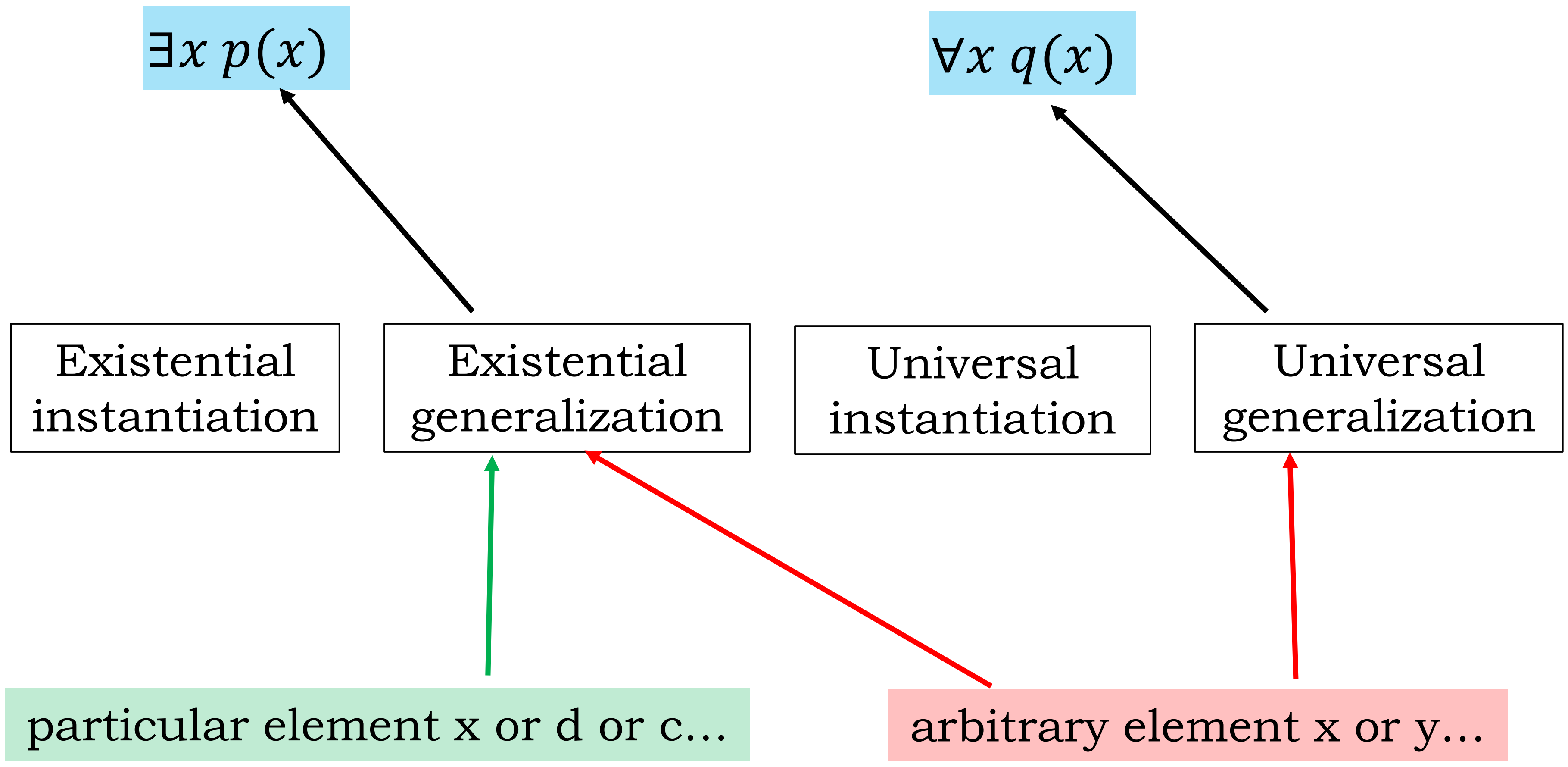
Universal generalization
(Adding a quantifier)

Arbitrary element means, it can be ***any*** element in the domain. So, we can pick any element and the statement is true for that element.

Rules of Inference with Quantifiers



Rules of Inference with Quantifiers



Proofs with Quantifiers - Example

A student Doug in the class knows how to program in JAVA. Everyone who knows how to program in JAVA can get a high paying job. Therefore, someone in the class can get a high paying job.

Proofs with Quantifiers - Example

A student Doug in the class knows how to program in JAVA. Everyone who knows how to program in JAVA can get a high paying job. Therefore, someone in the class can get a high paying job.

C(x): x is in the class.

J(x): x knows programming in JAVA

H(x): x can get high paying job

$$\underbrace{C(Doug) \land J(Doug) \land \forall x\, (J(x) \rightarrow H(x))}_{\text{Hypotheses}} \rightarrow \underbrace{\exists x\, (C(x) \land H(x))}_{\text{Conclusion}}$$

Proofs with Quantifiers - Example

$C(\text{Doug}) \wedge J(\text{Doug}) \wedge \forall x\, (J(x) \rightarrow H(x)) \rightarrow \exists x\, (C(x) \wedge H(x))$

1	$C(\text{Doug})$	Premise
2	$J(\text{Doug})$	Premise
3	$\forall x\, (J(x) \rightarrow H(x))$	Premise
4	$J(\text{Doug}) \rightarrow H(\text{Doug})$	3, Universal instantiation
5	$H(\text{Doug})$	2,4, Modus ponens
6	$C(\text{Doug}) \wedge H(\text{Doug})$	1,5, Conjunction
7	$\exists x\, (C(x) \wedge H(x))$	6, Existential generalization
8	QED	1 - 7

Proofs with Quantifiers – Another Example

$\forall x\,(P(x) \rightarrow (Q(x) \land S(x))) \land \forall x\,(P(x) \land R(x)) \rightarrow \forall x\,(R(x) \land S(x))$

1.	$\forall x\,(P(x) \rightarrow (Q(x) \land S(x)))$	Premise
2.	$\forall x\,(P(x) \land R(x))$	Premise
3.	$P(c) \land R(c)$	2, Universal instantiation
4.	$P(c)$	3, Simplification
5.	$P(c) \rightarrow (Q(c) \land S(c))$	1, Universal instantiation
6.	$Q(c) \land S(c)$	4,5, Modus ponens
7.	$S(c)$	6, Simplification
8.	$R(c)$	3, Simplification
9.	$R(c) \land S(c)$	7,8, Conjunction
10.	$\forall x\,(R(x) \land S(x))$	9, Universal generalization
11.	QED	1 - 10

Proofs with Quantifiers – Another Example

$\forall x\,(P(x) \rightarrow (Q(x) \land S(x))) \land \forall x\,(P(x) \land R(x)) \rightarrow \forall x\,(R(x) \land S(x))$

1.	$\forall x\,(P(x) \rightarrow (Q(x) \land S(x)))$	Premise
2.	$\forall x\,(P(x) \land R(x))$	Premise
3.	$P(c) \land R(c)$	2, Universal instantiation
4.	$P(c)$	3, Simplification
5.	$P(c) \rightarrow (Q(c) \land S(c))$	1, Universal instantiation
6.	$Q(c) \land S(c)$	4,5, Modus ponens
7.	$S(c)$	6, Simplification
8.	$R(c)$	3, Simplification
9.	$R(c) \land S(c)$	7,8, Conjunction
10.	$\forall x\,(R(x) \land S(x))$	9, Universal generalization
11.	QED	1 - 10

Pay attention to these steps.

Examples of Incorrect Proofs with Quantifiers

$$\exists x\, P(x) \;\wedge\; \exists x\, Q(x) \;\rightarrow\; \exists x\, (P(x) \wedge Q(x))$$

1.	$\exists x\, P(x)$	Hypothesis
2.	$\exists x\, Q(x)$	Hypothesis
3.	(c is a particular element) ∧ P(c)	Existential instantiation, 1
4.	(c is a particular element) ∧ Q(c)	Existential instantiation, 2
5.	P(c)	Simplification, 3
6.	Q(c)	Simplification, 5
7.	P(c) ∧ Q(c)	Conjunction, 5, 6
9.	$\exists x\, (P(x) \wedge Q(x))$	Existential generalization, 7
10	QED	

Is there any error in the argument?

Examples of Incorrect Proofs with Quantifiers

$\exists x\, P(x) \land \exists x\, Q(x) \rightarrow \exists x\, (P(x) \land Q(x))$

1.	$\exists x\, P(x)$	Hypothesis
2.	$\exists x\, Q(x)$	Hypothesis
3.	(c is a particular element) $\land$ P(c)	Existential instantiation, 1
4.	(c is a particular element) $\land$ Q(c)	Existential instantiation, 2
5.	P(c)	Simplification, 3
6.	Q(c)	Simplification, 5
7.	$P(c) \land Q(c)$	Conjunction, 5, 6
9.	$\exists x\, (P(x) \land Q(x))$	Existential generalization, 7
10	QED	

The value of x for which P is true might be **different** than the value of x for which Q is true. But, we have assumed that it's the same, that is c for both the cases.

Incorrect Proofs with Quantifiers

Useful Tip:

Be Careful in Using Existential Instantiation

$\exists x$ P(x) means that there exists *"some"* value of x which for which P(x) is true. We cannot just pick the value of our choice and say P is true for that value.

Examples of Incorrect Proofs with Quantifiers

Rachel is taking discrete math. Rachel is a computer science major student. Therefore, every computer science major student takes discrete math.

$D(x)$: x is taking discrete math.

$C(x)$: x is a computer science major student

$$D(\text{Rachel}) \land C(\text{Rachel}) \rightarrow \forall x\,(D(x) \land C(x))$$

1.	D(Rachel)	Premise
2.	C(Rachel)	Premise
3.	$D(\text{Rachel}) \land C(\text{Rachel})$	1,2, Conjunction
4.	$\forall x\,(D(x) \land C(x))$	3, Universal generalization
5	QED	1 - 4

Examples of Incorrect Proofs with Quantifiers

Rachel is taking discrete math. Rachel is a computer science major student. Therefore, every computer science major student takes discrete math.

D(x): x is taking discrete math.

C(x): x is a computer science major student

$$D(Rachel) \land C(Rachel) \rightarrow \forall x\, (D(x) \land C(x))$$

1.	D(Rachel)	Premise
2.	C(Rachel)	Premise
3.	D(Rachel) $\land$ C(Rachel)	1,2, Conjunction
4.	$\forall x\, (D(x) \land C(x))$	3, Universal generalization
5	QED	1 - 4

Incorrect argument

Incorrect Proofs with Quantifiers

Useful Tip:

Be Careful in Using Universal Generalization

If statement is true for a *particular* value of variable, then it does not mean that it is necessarily true for *all* values of variables. In other words, be careful about stereotyping. If you're going to apply "for all" to something, you better be sure it's applicable to "all."