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Research Article

Necessary and Sufficient conditions for Normality of Operators in Hilbert spaces

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Abstract

Characterization of normality is an interesting aspect for Hilbert space operators. In this paper, we have shown that for an operator A to be normal, it is necessary that A = A*. It is also sufficient that for an operator A to be normal then the condition AA* = A*A holds. Moreover, for an inner derivation, we conjecture that the property $^{\delta}{}_{A}{=}^{\delta}{}_{A*}$ is necessary for its normality.

Keywords: Adjoint Operator; Normal operators; Posinormal operators; Positive operators.

Introduction

The field of analysis has been very interesting especially on the study of elementary operators for many decades. Sylvester in 1880s [1], computed the eigenvalues of the matrix operators on a square matrix. This work has been of great concern especially in the applications of operator theory and functional analysis. Later, Lumer and Rosenblum [2] described the elementary operator from a mapping T : A → A if it can be expressed as T : B(H) → B(H) by $T_{Ai,Bi}(X) = \sum^{n}_{i=1} A_i X B_i$ ∀ X ∈ B(H) and ∀ A_i, B_i fixed in B(H) and 1 ≤ i < n. The study of operator theory has been significant dating back many decades ago [3].

Some research has been done though not exhaustive. Studies about elementary operators have been of much concern. We define an elementary operator T : B(H) → B(H) [6] by $T_{Ai,Bi}(X) = \sum^{n}_{i=1} A_i X B_i$ ∀ X ∈ B(H) and ∀ A_i, B_i fixed in B(H) where i = 1, . . . , n [4]. From this operator, we can define the generalized adjoint by $T_{Ai,Bi}(X) = \sum^{n}_{i=1} A_i^* X B_i^*$ and we say that T is normal if and only if T T*= T*T. Now AC = CA, BD = DB, together with AA*= A*A, BB*= B*B, CC*= C*C and DD*= D*D ensures that the operator $T_{Ai,Bi}(X)$ = AXC + BXD is normal [5]. Some of our results show that; if T ∈ B(H) be a p-hyponormal and T = U |T | be polar decomposition of T such that U^{n0}= U* for some positive integer n0 then T is normal. Moreover, if T ∈ B(H) be a p-hyponormal ant T = U |T | be the polar decomposition of T such that $U^{*}n$→1 or Un→ 1 as n → ∞, where limits are taken in the strong operator topology then T is normal [6]. For an operator A to be normal, it is also necessary that A = A*. It is sufficient that for an operator A to be normal then the condition AA* = A*A holds [7]. This knowledge is important especially in quantum physics especially the formulation of Heisenberg uncertainty principle for linear transformations and non-zero scalars such that AX − XA = αI [8]. The study can also be used in the solutions of Schrondinger wave equations since the infimum of the Hamiltonian operator is always an eigenvalue and its corresponding eigenvector are called the ground state energies E giving us a formulation of E as ($E_{C3, H8}$) [9].

Over the past years, several scholars have joined in research to describe several properties related to the structure of the elementary operators. Rodman [10] described Sylvester and Lyapunov operators in real and complex matrices which included in particular cases operators arising from the theory of linear time invariant system. Fanqyan [11] described the multiplicative mappings of operator algebras. They described the nest algebra as being the natural analogues of upper triangular matrix algebra in the infinite dimensional Hilbert space. Gheondea [12], described the normality of elementary operators based on the spectral theorem for the normal operators. This study Postulated that If N ∈ B(H) is a normal arbitrary

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such that AN = NA then $AN^* = N^*A$ as well is normal. This shows that that if A, B ∈ B(H) are two normal operators that commute and each commutes with its adjoint, then their product is AB is normal [13]. The study further deduces that if A and B are bounded operators such that AB is normal and compact, then BA is normal and compact as well and sk(AB) = sk(BA) for all k = 1, 2, . . .n [14] The objective of this study was to determine the necessary and sufficient conditions for normality of Hilbert space operators. These conditions have been obtained for Hilbert space operators and a conjecture given for inner derivations.

Research methodology

Here we define some of the key terms and give some basic concepts that are used in our work.

Definition 1.1. ([15], Definition 1.2.1) Field. A field F is a set closed under two binary operations of addition and scalar multiplication satisfying the following properties:

(i). Closure under addition and multiplication. a + b ∈ F and a.b ∈ F, ∀ a, b ∈ F,
(ii). Associativity: a + (b + c) = (a + b) + c, ∀ a, b, c ∈ F,
(iii). commutativity: a + b = b + a and (a.b).c = (b.c).a, ∀ a, b, c ∈ F,
(iv). Additive and multiplicative identities: ∀ a ∈ F, ∃ − a ∈ F: a + −a = 0. And ∃ a^{-1}∈ F: $a.a^{-1}=1$
(v). Distributivity: a(b + c) = (ab + ac) ∀ a, b, c ∈ F,
(vi). Existence of additive inverse: ∀ a ∈ F ∃ x ∈ K: a + x = 0, and x + a = 0 then a = −x ∀a, x ∈ F,
(vii). Existence of a multiplicative inverses: For each a ∈ F with 0<a> 0 the equations a.x = 1 and x.a = 1 have a solution x ∈ F, called the multiplicative inverse of a and denoted by a^{-1}.

Definition 1.2. ([16], Definition 1.1.2) Vector space. Let F be a field and V a collection of objects called vectors, then V is a vector space over a field F if V is closed under vector addition and scalar multiplication. i.e. ∀ v_1, v_2 ∈ V, $v_1 + v_2$ ∈ V and ∀ v ∈ V, and ∀ a ∈ F, a.v ∈V , and satisfies the following properties:
(i). Commutativity. $v_1 + v_2 = v_2 + v_1$, ∀ v_1, v_2 ∈ V,
(ii). Associativity. $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$. ∀ v_1, v_2, v_3 ∈ V,
(iii). Additive inverse. ∀ v ∈ v, ∃ − v ∈ V : v + −v = 0 ∀ v_1, − v ∈ V
(iv). Additive Identity. ∀ v ∈ V, ∃ 0 ∈ V : v + 0 = v. ∀ v ∈ V
(v). Multiplicative Identity. 1.v = v, ∀ v ∈ V
(vi). Distributive property. ∀ a ∈ F, and ∀ v_1, v_2 ∈ V, $a(v_1 + v_2) = (av_1 + av_2)$ and the space (V, ‖.‖) is called a normed vector space.
(vii). Unitary law. ∀ v ∈ V, 1.v = v.

Definition 1.3. ([17], Definition 2.1.8) Banach space. This is a complete normed linear space.

Definition 1.4. ([18], Definition 2.7) Hilbert space. A Hilbert space is a complete inner product space.

Definition 1.5. ([19], Definition 2.1.8) Norm. A norm is a non-negative real valued function that takes the elements of a vector space to a field of real numbers denoted by ‖.‖: V → R satisfying the following conditions:
(i.) Non-negativity: ‖x‖ ≥ 0, ∀ x ∈ V.
(ii.) Zero property: ‖x‖ = 0, if and only if x=0, for all x ∈.
(iii.) Homogeneity: ‖αx‖ ≤ |α|‖x‖, ∀ x ∈ V and α ∈ F
(iv.) Triangle inequality: ‖x + y‖ ≤ ‖x‖ + ‖y‖, ∀ x and y ∈ V
The pair (V, ‖.‖) is called a normed linear space.

Definition 1.6. [7]. Elementary Operator. Let H be an infinite dimensional complex Hilbert space and B(H) be an algebra of all bounded linear operators on the H . We define an elementary operator T : B(H) → B(H) by $T_{A_i,B_i}(X) = \sum_{i=1}^{n} A_i X B_i$ ∀ X ∈ B(H) and ∀ A_i , B_i fixed in B(H) where i = 1, . . . , n. Examples of elementary operators include:
(i). The left multiplication operator L_A: B(H) by: $L_A(X) = AX$, ∀X ∈ B(H).
(ii). The right multiplication operator R_B: B(H) by: $R_B(X)=BX$, ∀X ∈ B(H).
(iii). The Basic elementary operator (implemented by A, B) by: $M_{A,B}(H) = AXB$, ∀X ∈ B(H).
(iv). The Jordan elementary operator (implemented by A, B) by: $U_{A,B}(X)=AXB + BXA$, ∀X ∈B(H).
(v). The Generalized derivation (implemented by A, B) by: $\delta_{A,B} = L_A - R_B$.
(vi).The inner derivation (implemented by A, B) by: $\delta_A = AX – XA$.

Definition 1.7. ([3] Definition 1.3) A normal operator. Let T ∈ B(H) and T^* ∈ B(H).

Then T is said to be normal if and only if $TT^* = T^*T$.

Definition 1.8. ([1] Definition 1.8) Adjoint of an operator. Let A be a bounded linear operator on a Hilbert space H. The operator A^*: H → H defined by (Ax, y) = (x, A^*Y) for all y ∈ H, is called the adjoint of the operator A.

Definition 1.9. ([12], Definition 1.) Hyponormal Operators. Let H be a Hilbert space and T ∈ B(H) then we say that T is hyponormal if ‖T x‖ = ‖T*x‖ i.e. T*T − T T* = 0 for all x ∈ H

Definition 1.10. ([8], Definition 1.) p-hyponormal operator. Let H be a Hilbert space and T ∈ B(H) then we say that T is p-hyponormal $0 < p \leq 1$ if (T T*)p ≥ (T T*)p where T* is the adjoint of T .

Definition 1.11. ([13], Definition 7.) Invertible operator. Let H be a Hilbert space and T an operator in H, then T is said to be invertible if there exists T^{-1} called the inverse of T such that $T^{-1}T = TT^{-1} = I$.

Definition 1.12. ([9], Definition 1.) Quasinilipotent operator. Let H be a Hilbert space and T be an operator such that T ∈ B(H) and σ(T) be the spectrum of T . We say that T is quasinilipotent if σ(T) = 0

Definition 1.13. ([13], Definition 1.) Positive operator. Let H be a Hilbert space and T ∈ B(H), then we say that T is positive if ⟨Tu, v⟩ ≥ 0.

Definition 1.26. ([13], Definition 2.) Skew - Hermitian operator. Let H be a Hilbert space and T ∈ B(H) then T is Hermitian if T*= −T.

Results and discussion

An adjoint of a bounded linear operator T is also linear, bounded and unique. This can be shown by the result below.

Proposition 1. Let (Y, K) be Hilbert spaces and T ∈ B(Y, K) then there exists a unique bounded linear operator T^*∈ B (K, Y) such that; ⟨T x, y⟩ = ⟨x, T^*y⟩ for all x ∈ Y and y ∈ K and ‖T ‖ = ‖T^*‖ (i.e. T^* is an adjoint of T, $(T^*)^*$= T ∈ B(H)).

Proof. Let y ∈ K be arbitrary and ∀x ∈ Y, we define $f_y(x)$ = ⟨T x, y⟩ ∀ x ∈ Y. We need to show that $f_y \in Y^*$ and that f_y is linear and bounded. Let x, x′ ∈ Y: λ, λ′∈ C then; f_y (λx + λ′x′) = ⟨T (λx + λ′x′)y⟩ = ⟨λTx + λ′Tx′y⟩ = λ⟨T x, y⟩ + λ′⟨T x′, y ⟩ = $\lambda f_y(x) + \lambda' f_y(x')$.

Hence f_y is linear.
To show boundedness we have:
|fy(x)| = |⟨T x, y⟩| ≤ ‖T x‖‖y‖, by CBS, ≤ ‖T ‖ ‖x‖‖y‖.
Therefore, ‖fy‖ ≤ ‖T x‖ ‖y‖ hence bounded.
By Riez's representation theorem, f (x) = ⟨x, y^*⟩ for some unique y^*∈ Y and ‖fy‖ = ‖y^*‖. For y ∈ K, we have a unique y^*∈ Y. This helps us to define T^*: K → Y by $T^*(y) = y^*$ then we claim that T^* is linear. Let $y_1, y_2 \in K$ and $\beta_1, \beta_2 \in C$, we can re-write; ‖fy‖ = ‖T^*y‖.
Now, $f_{\beta_1 y_1, \beta_2, y_2}(x)$ = ⟨Tx, β_1, $y_1+\beta_2$, y_2⟩ = ⟨x, T^*(β_1,$y_1+\beta_2 y_2$)⟩.
But, ⟨T x, β_1,$y_1+\beta_2y_2$⟩ = ⟨T x, β_1,y_1⟩+ ⟨T x, β_2y_2⟩= β_1⟨Tx,y_1⟩+β_2⟨Tx, y_2⟩= β_1⟨x, T^*y_1⟩ + β_2⟨x, T^*y_2⟩= ⟨x, $\beta_1T^*y_1$⟩ +⟨x,β_2T^*y2⟩=⟨x,$\beta_1T^*y_1$+ $\beta_2T^*y_2$⟩.
Therefore, ⟨x, $\beta_1T^*y_1$+ $\beta_2T^*y_2$⟩= ⟨x, T^*(β_1,$y_1+\beta_2y_2$)⟩ for all x ∈ Y. Hence T^*(β_1,$y_1+\beta_2y_2$) = $\beta_1T^*(y_1)$ + $\beta_2T^*(y_2)$ i.e. T^* is linear. If T^* ∈ B (K, Y), then we have that ‖ T^*y‖ ≤ ‖T ‖‖y‖ i.e. T^* is bounded and ‖T^*‖≤‖T ‖. It is clear now that T^* is unique (for some unique y ∈ Y). Since T^* ∈ B(K, Y), we apply the above reasoning to obtain its adjoint $(T^*)^*$∈ B(Y, K) and we have that; ⟨ T^* y, x⟩ = ⟨y, T^{**} x⟩, ∀ Y ∈ K and x ∈ Y and ⟨T*y, x⟩ = ⟨x, T*y⟩ = ⟨T x, y⟩ = ⟨y, T x⟩. We now show that ‖T^{**}‖ ≤ ‖T*‖. So we have that, ⟨y, T^{**}x⟩ =⟨y, T x⟩ ∀ y ∈ K and x ∈ Y i.e. ⟨y, T**x − T x⟩ = 0, i.e. T**x = Tx hence ⇒ T**= T thus ‖T*‖ ≤ ‖T ‖, ‖T**‖ ≤ ‖T*‖ and ‖T**‖ = ‖T ‖ so ‖T*‖ ≤ ‖T ‖ and ‖T ‖ ≤ ‖T*‖ hence ‖T*‖ = ‖T ‖

Proposition 2. If A ∈ B(H) and ⟨Ax, x⟩ = 0, ∀ x, y ∈ H, then A = 0.
Proof. Let x, y ∈ H, then; ⟨A(x + y), x + y⟩ = ⟨Ax + Ay, x + y⟩ = ⟨Ax, x⟩ + ⟨Ax, y⟩ + ⟨Ay, x⟩ + ⟨Ay, y⟩ ……………………………………(1)
⟨A(x − y), x − y⟩ = ⟨Ax − Ay, x − y⟩ = ⟨Ax, x⟩ − ⟨Ax − y⟩ − ⟨Ay − y⟩ + ⟨Ay, y⟩ ……………(2)
⟨A(x + y), x + y⟩ = ⟨Ax + iAy, x + iy ⟩……..(3)
= ⟨Ax, x⟩ + i⟨Ax, y⟩ + i⟨Ay, x⟩ + ⟨Ay, y⟩....(4)
⟨A(x − y), x − y⟩ = ⟨Ax − iAy, x − iy ⟩………(5)
= ⟨Ax, x⟩ − ⟨Ax, y⟩ − i⟨Ay, x⟩ + ⟨Ay, y⟩…..(6)
Subtracting (2) from (1) gives; 2⟨Ax, y⟩ + 2⟨Ay, x⟩. Subtracting i × (6) from (4) gives; 2⟨Ax, y⟩ − 2⟨Ay, x⟩. Adding, 2⟨Ax, y⟩ + 2⟨Ay, x⟩ + 2⟨Ax, y⟩ − 2⟨Ay, x⟩ = 4⟨Ax + y⟩.
Thus, ⟨Ax, y⟩ = $\frac{1}{2}${⟨A(x + y), x + y⟩ −⟨A(x − y), x − y⟩ + i⟨(x + iy), x +iy⟩ − i⟨A(x − iy), x − iy⟩}, ∀ x, y ∈ H. Since ⟨Ax, x⟩ = 0, ∀x, y ∈ H, the right hand side of the equation is zero.

i.e. ⟨Ax, y⟩ = 0, ⇒ Ax⊥Y ⇒ Ax¯0 ⇒ A = 0.

Proposition 3. Let A ∈ B(H), then it is sufficient that A is normal if and only if it commutes with its adjoint A* i.e. A*Ax = AA*x for all x ∈ H thus A ∈ B(H) is normal if and only if ‖Ax‖ = ‖A*x‖ and that AA*= A*A.

Proof. To see this, let A ∈ B(H) be normal, i.e. A = A* thus A*Ax =AA*x ∀ x ∈ H then; ⟨A*Ax, x⟩ = ⟨AA*x, x⟩ ∀x ∈ H i.e. ⟨Ax, Ax⟩ =⟨A*x, A*x⟩ ∀ x ∈ H. i.e. ‖Ax‖2= ‖A*x‖2⇒ ‖Ax‖ = ‖A*x‖, ∀ x ∈ H⇒ ‖A‖‖x‖ = ‖A*‖‖x‖⇒‖A‖= ‖A*‖ hence A = A*. Conversely, Let ‖Ax‖ = ‖A*x‖ ∀x ∈ H, i.e. ⟨A*Ax, x⟩ = ⟨AA*x, x⟩ ∀x ∈ H, i.e. ⟨ (A*A − AA*) x, x⟩ = 0 ∀ x ∈ H. And that (A*A − AA*) x = 0 ∀ x ∈ H. It follows that by proposition [2] that A*A – AA*= 0 i.e. A*A = AA*.

Theorem 4. Let A, B, X ∈ B(H) such that A* is p-hyponormal, B is dominant and X is invertible, if AX = BX , then there exists a unitary U such that AU = UB and hence A and B are normal.

Proof. Since AX = BX, then it follows by Fuglede-Putnam theorem that for p-hyponormal ([16], Theorem 2) B∗X = XA* and so X* B = AX*. Now, AX* X = X* BX = X* XA. Let X = UP be polar decomposition of X. Since X is invertible, it follows that P is invertible and U is unitary. Since $AP^2= P^2$ and P is positive, it follows that AP = PA. Thus BUP = UPA ⇒ BUP = UAP. But P is an invertible so we have BU = UA. Therefore, A and B are unitarily equivalent. So, A is dominant and B*is p-hyponormal. Hence A, B are normal.

Theorem 5. Let T = A + iB ∈ B(H) be Cartesian decomposition of T with AB is p-hyponormal. If A or B is positive, then T is normal.

Proof. Assume that A is positive, Let S = AB then SA = AS∗. Then it follows that from Fugled-Putnam theorem for p-hyponormal ([16], Theorem 2) that S*A = AS, that is $BA^2= A^2B$. But B is positive, then AB = BA hence T is normal.

Theorem 6. Let B be a bounded normal operator. Let A be an unbounded normal operator. Assume that B commutes with A. If for some r > 0, ‖rBB*− I ‖ < 1, then BA is normal.

Proof. We need to show the closedness of BA. Let xn → x and BAxn →y, then the condition ‖rBB*− I ‖ < 1 plus the normality of B guarantees that BB*= B*B is invertible. Hence by continuity of B∗, B*BAxn →B*y. Therefore, AXn → (B*B) −1B*y. This implies that B*BAx =B*y and hence BB*BAx = BB*y. With invertibility of BB*, we have that BAx = y proving the closedness of BA.

Theorem 7. Let A, V, X ∈ B(H) be such that V, X are isometries and A* is p-hyponormal.
If VX = XA, then A is normal.

Proof. Since V X = XA, then by Fugled-Putnam theorem, we have that V*X = XA*. Multiplying V X = XA by V*, we get X = V*XA, then X (I – AA*) = 0 implies that X*X (I – A*A = 0.) So A is an isometry. Therefore A and A* are p-hyponormal and hence A is a normal isometry.

Theorem 8. Let A, B ∈ B(H) be such that A and AB are normal. Then BA is normal if and only if B commutes with |A|.

Proof. Since A = U |A|, where U ∈ B(H) is unitary and commutes with |A| = √(A∗A) , if in addition B commutes with |A|, then U*ABU = U*U |A|BU = B|A| = BU |A| = BA and hence BA is normal as well (as unitary operator with the normal operator AB.) conversely, if BA is normal, let M = AB and N = BA. Then MA = ABA = AN. By Fuglede-Putnam theorem, it follows that M*A = AN*, that is, B*A*A = AA*B* and taking into account that A*A = AA* this means that B* commutes with A*A and so B.

Theorem 9. Let T = A + iB be the Cartesian decomposition of T. If T* is hyponormal and AB is p-hyponormal, then T is a normal operator.

Proof. Let Q = AB, then QA = AQ∗= ABA. Then by Fuglede-Putnams theorem, we have that Q∗A = AQ i.e $BA^2= A^2B$. Now, (Q + Q*) A =A(Q + Q*) and (Q − Q*)A = A(Q − Q*) Since T* is hyponormal, we have that T T*−T*T = 2i(BA−AB) = 2i(Q∗−Q) ≥ 0. Let Y = 2i(BA−AB) then;
Y ≥ 0 and Y A = −AY.
Now Y^2A = Y (Y A) = Y (−AY) = −Y AY = − (−AY) Y = AY^2.

But Y is positive, then Y A = AY = 0. Hence, A(AB − BA) = (AB − BA)A = 0 implies that σ(AB − BA) = 0 therefore AB − BA is quasinilipotent skew Hermitian. Thus AB − BA = 0 so T is normal.

Theorem 10. Let T ∈ B(H) be p-hyponormal and T = U |T | be polar decomposition of T such that U^{n0}= U* for some positive integer n_0 then T is normal.

Proof. Let T be p-hyponormal for some p > 0. Hence |T |2p≥ |T∗|2p=U |T |2pU∗. Multiplying both sides of the inequality ($|T|^{2p} \geq |T^*|^{2p}$) by U and U* we have that $U |T|^{2p} U^* \geq U^2|T|^{2p} U^{2*}$ hence $|T|^{2p} \geq U |T|^{2p} U^{2p*}$. Repeating this process we have the inequalities: $|T|^{2p} \geq |T^*|V= U |T|^2U^{2p} \geq U^2|T|^{2p} U^{2p*} \geq \ldots \geq U^{n0}|T|^{2p} U^{n0+1}$................................. (4.2.16) Since $U^{n0}= U^*$, we observe that $U^{n0+1}= U^*U = U^{(n0+1)*}$ is a projection onto Ran|T| hence, $U^{n0+1}|T|^{2p} U^{(n0+1)*}= |T|^{2p}$ from which and inequality, [4.2.16], we obtain $|T|^{2p} \geq |T*|^{2p}$ thus $|T|^2= |T^*|^2$ hence normal.

Theorem 11. Let T ∈ B(H) be satisfying the following conditions:
(i.) T is a restriction-Convexoid
(ii.) T is reduced by each of its eigenspaces
(iii.) T = S−1ApS + K where σ (A) is real, K is compact and p is some non-negative integer. Then T is normal
Proof. By Weyl's spectrum we have $\sigma_w(T) = \sigma(T) - \sigma_{00}(T)$. Since Weyl's spectrum is preserved under similarity and also remains invariant under compact perturbation, we have $\sigma_w(S−1ApS + K) = \sigma(S−1ApS) =\sigma_w(Ap) \subseteq \sigma(A)p$. So $\sigma_w(T)$ is real. Let $T_1 = T \setminus H$, be the restriction of T to the subspaces H_1 generated by eigenvectors corresponding to eigenvalues, $\lambda_0 \in \sigma_{00}(T)$. Let $H_2 = H_1\perp$ and $T_2 = T \setminus H_2$, then we obtain subspaces $H_1 \oplus H_2$. Since T is reduced by each of its eigenspaces, we conclude that T is normal. Also $\sigma(T_2) = \sigma_w(T)$ is real and hence T_2 is self adjoint which shows that T is normal.

Theorem 12. Let T ∈ B(H) be a p-hyponormal ant T = U |T | be the polar decomposition of T such that $U^{p\,*\,n} \rightarrow 1$or Un→ 1 as n → ∞ where limits are taken in the strong operator topology. Then T is normal.
Proof. Let $U^{*n} \xi \rightarrow \xi$ as $n \rightarrow \infty$ ∀ ξ ∈ H. In this case, Un→ 1 in the strong operator topology then it follows by inequalities, [4.2.16], that
$\||T|^p \xi\| \geq \|T^*|^p \xi\| = \||T|^p U \xi\| \geq \||T|^p U^p{}^*\xi\| \geq \ldots \||T|^p \xi\| \geq \||T|^p U^n \xi\| \geq \ldots$.(4.2.17)
Since$| \||T|^p U^{*n} \xi\| - \||T|^p \xi\| | \leq \||T|^p U^n \xi\| - |T|^p \xi\| \leq \||T|^p\|\|U^{*n} \xi - \xi\| \rightarrow 0$ as $n \rightarrow \infty$ we have that $\|T|^p U^{*n} \xi\| \rightarrow \||T|^p \xi\|$ as $n \rightarrow \infty$ hence by inequalities, [4.2.17], we get $\||T|^p \xi\|^2=\||T^*|\xi\|^2$, so $|T|^{2p} = |T^*|^{2p}$ hence T is normal.

Conclusions

The structural properties of the elementary operators have been of great concern in analysis mathematics. Several of properties have been studied and of the most interesting concern is the norm property. The term elementary operator came as a result of the knowledge of the basic elementary operators from an algebra. If A is an algebra, then given a, b ∈ A, we define the basic elementary operator (implemented by A, B) by: $M_{A,B}(H) = AXB$, ∀ X ∈ B(H). This led to the form describing the elementary operators as the sum of basic elementary operators. T : B(H) → B(H) by $T_{Ai,Bi}(X) =\sum^n_{i=1}A_i X B_i$ ∀ X ∈ B(H) and ∀ A_i , B_i fixed in B(H). For this operator A to be normal, it is necessary that A = A*. It is also sufficient that for an operator A to be normal then the condition AA* = A*A holds. He normality question has not been exhausted. For example, for an inner derivation operator, we conjecture that the property $\delta_A=\delta_{A^*}$.

Conflicts of Interest

The authors hereby declare that they have no conflict of interest.

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