Math 1496 Calc I

Areas Under Curves

Before doing so we want to introduce sums & sigma notation

Consider $1+2+3+4+\cdots 10$

we can conveniently write this as

$$\sum_{i=1}^{10} i$$

i - generator

$i=1 \to 10$ index & Σ sums

So $\sum_{i=2}^{5} i^2 = 2^2+3^2+4^2+5^2$

$$\sum_{j=3}^{6} \frac{1}{\sqrt{j}} = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}}$$

in general $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots a_n$

Similarly if we are given the sum

$$(2^2-1)+(3^2-1)+(4^2-1)+(5^2-1)+(6^2-1)+(7^2-1)$$

then $\sum_{i=2}^{7} i^2-1$

Some properties

(1) $\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i$

(2) $\sum_{i=1}^{n} a_i \pm b_i = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Formulas

we have some formulas for some standard sums

(1) $\sum_{i=1}^{n} 1 = \underbrace{1+1+1+1+\cdots 1}_{\text{there are } n \text{ of these}}$

$= n$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof

$$S = \sum i = 1 + 2 + 3 + 4 + \overset{(n-1)}{\cdots} \; n$$

$$= n + (n-1) + (n-2) + \cdots \; 2 + 1$$

$$2S = \underbrace{(n+1) + (n+1) + (n+1) \cdots}_{n \text{ of these}} (n+1)$$

$$2S = n(n+1)$$

$$\Rightarrow \quad S = \frac{n(n+1)}{2}$$

others

$$\sum i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum i^3 = \frac{n^2(n+1)^2}{4}$$

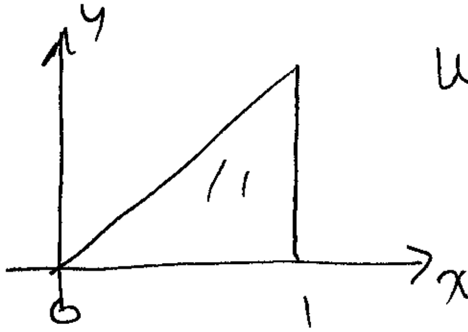
So

$$\sum_{i=1}^{n} 2i^2 - 3i + 4 = 2\sum i^2 - 3\sum i + 4\sum 1$$

$$= \frac{2n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + 4 \cdot n$$

$$= \frac{4n^3 - 3n^2 + 17n}{6}$$

Areas Consider $f(x) = x$ on $[0,1]$



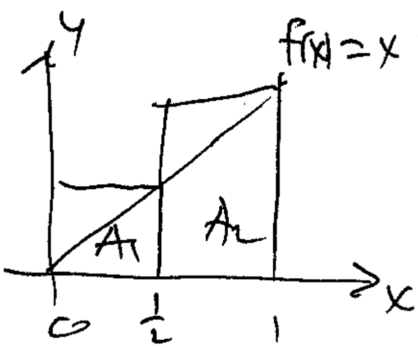
we know the area is $\frac{1}{2}$

$(\frac{1}{2} b h)$

However if we consider $f(x) = x^2$

this area we don't know.

One idea is to approx. the area with rectangles

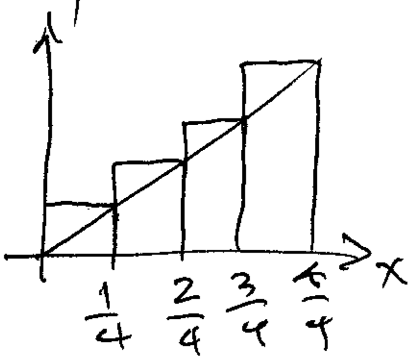


If we use 2 of equal width then

$A_1 = \frac{1}{2}(\frac{1}{2})$ $A_2 = \frac{1}{2}(1)$

$A \approx \frac{1}{4} + \frac{2}{4} = \frac{3}{4}$ too much

Next 4 rectangles



$A_1 = \frac{1}{4}\left(\frac{1}{4}\right)$

$A_2 = \frac{1}{4}\left(\frac{2}{4}\right)$

$A_3 = \frac{1}{4}\left(\frac{3}{4}\right)$

$A_4 = \frac{1}{4}\left(\frac{4}{4}\right)$

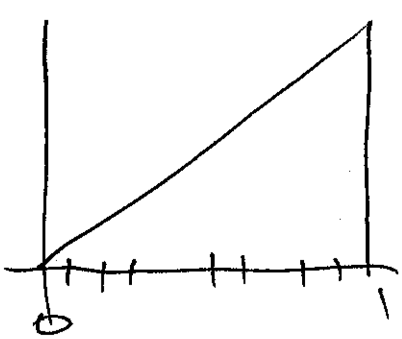
$$A \approx \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{2}{4}\right) + \frac{1}{4}\left(\frac{3}{4}\right) + \frac{1}{4}\left(\frac{4}{4}\right)$$

$$= \frac{1+2+3+4}{16} = \frac{10}{16} = \frac{5}{8} = \frac{1}{2} + \frac{1}{4}$$

we certainly could continue in this fashion

8, 16, 32 etc rectangles

instead we will consider n rectangles of equal width (thickness)



if the interval is $[0,1]$ then the thickness of each rectangle (we will call this Δx)

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

so $A_1 = \frac{1}{n}\left(\frac{1}{n}\right)$ $A_2 = \frac{1}{n}\left(\frac{2}{n}\right)$ $A_3 = \frac{1}{n}\left(\frac{3}{n}\right)$

--- $A_i = \frac{1}{n}\left(\frac{i}{n}\right)$ --- $A_n = \frac{1}{n}\left(\frac{n}{n}\right)$

$$A \simeq \frac{1}{n}\left(\frac{1}{n}\right) + \frac{1}{n}\left(\frac{2}{n}\right) + \cdots \frac{1}{n}\left(\frac{i}{n}\right) + \cdots \frac{1}{n}\left(\frac{n}{n}\right)$$

ith rectangle

$$= \frac{\sum_{i=1}^{n} i}{n^2}$$

Now $\sum i = \frac{n(n+1)}{2}$

$$A = \frac{n^2+n}{2n^2} = \frac{1}{2} + \frac{1}{2n}$$

Now limits

$$A = \lim_{n \to \infty} \frac{1}{2} + \frac{1}{2n} = \frac{1}{2}$$ (which we know)