Calculus 3 - Vector Functions

We were introduced to vectors in Calc 2. Simply put, a vector is a directed line segment. For example, consider the points $P(1,1)$ and $Q(2,3)$. The line connecting $P \to Q$ is the vector (see figure 1). Note we have an arrow to denote it has direction.

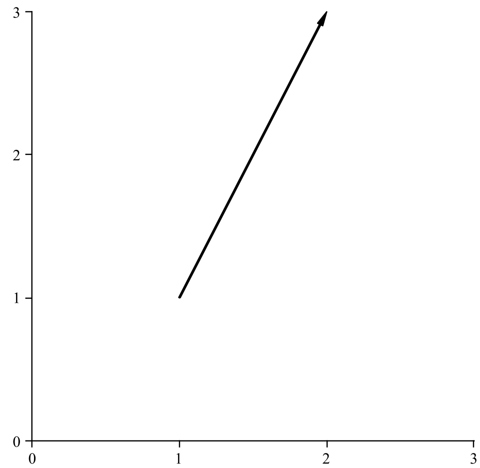


Figure 1: A vector

Symbolically

The vector in this case is $\vec{u} = \overrightarrow{PQ} =< 2-1, 3-1 >=< 1, 2 >$. The magnitude is

$$\|\vec{u}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Vector Functions

Now we extend the idea of vectors and allow the components to vary and in this case with respect to t. So we define a vector function as

$$\vec{r}(t) =< f(t), g(t) >$$

For example

$$(i) \quad \vec{r}(t) = < t, t+1 >$$
$$(ii) \quad \vec{r}(t) = < t, \tfrac{1}{4}t^2 + 1 >$$
$$(iii) \quad \vec{r}(t) = < 2\cos(t), 2\sin(t) >$$

As we did with functions, we can create a table of values. So in the first case

t	$\vec{r}(t) = < t, t+1 >$
-2	$< -2, -1 >$
-1	$< -1, 0 >$
0	$< 0, 1 >$
1	$< 1, 2 >$
2	$< 2, 3 >$

and then we draw each one

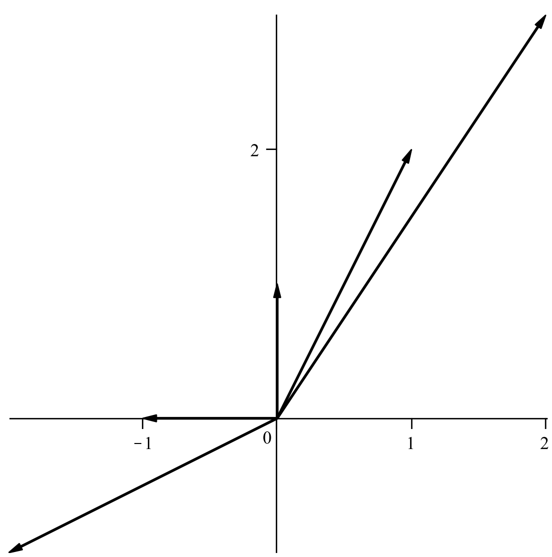


Figure 2: Vector Function $\vec{r}(t) = < t, t+1 >$

Similarly, with (ii) and (iii) we see

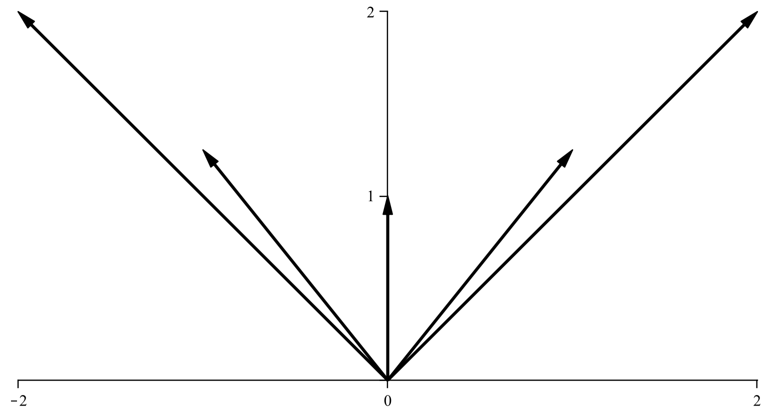


Figure 3: Vector Function $\vec{r}(t) = < t, \frac{1}{4}t^2 + 1 >$

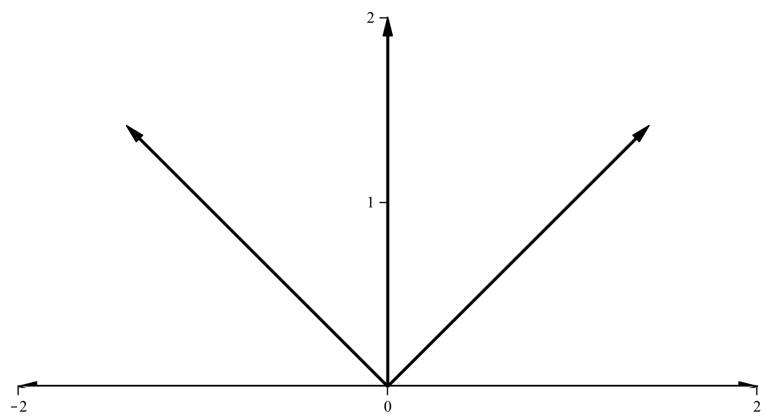


Figure 4: Vector Function $\vec{r}(t) = < 2\cos(t), 2\sin(t) >$

As we sketch the vector function, we see a curve is traced out. In example (i) we see that the curve is given by

$$x = t, \quad y = t + 1 \tag{1}$$

and eliminating t gives

$$y = x + 1 \tag{2}$$

Figure 5: Vector Function(i)

Similarly in (ii)

$$x = t, \quad y = \frac{1}{4}t^2 + 1, \quad \Rightarrow \quad y = \frac{1}{4}x^2 + 1 \tag{3}$$

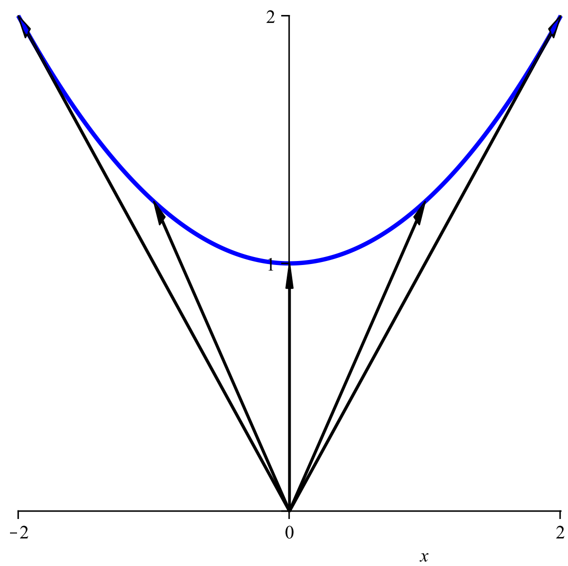


Figure 6: Vector Function (ii)

and (iii)

$$x = 2\cos(t), \quad y = 2\sin(t), \quad \Rightarrow \quad x^2 + y^2 = 4 \tag{4}$$

What's important to recognize is that as the vector varies, the space curve is drawn with direction.

In general, if the vector functions is

$$\vec{r} = < f(t), g(t) > \tag{5}$$

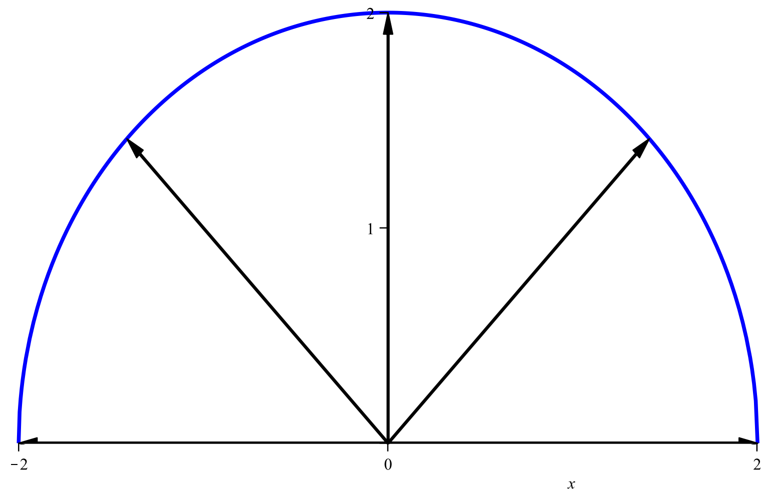


Figure 7: Vector Function (iii)

the space curve is

$$x = f(t), \quad y = g(t) \tag{6}$$

3D Vector Functions

This easily extends to $3D$. So we have

$$\vec{r} =< f(t), g(t), h(t) > \tag{7}$$

and the space curve

$$x = f(t), \quad y = g(t), \quad z = h(t) \tag{8}$$

For example consider

$$\vec{r} =< \cos(t), \sin(t), t > \tag{9}$$

The space curve here is

$$x = \cos(t), \quad y = \sin(t), \quad z = t. \tag{10}$$

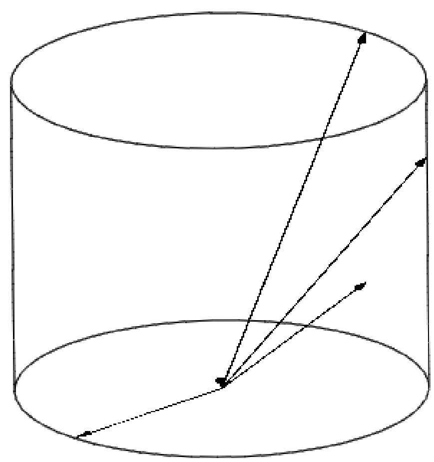


Figure 8: 3D Vector Function

Calculus of Vector Functions

With the introduction of vector functions, we now explore the calculus of vector functions. So the ideas introduced in Calculus 1 such as limits, continuity, derivatives and integration also apply here.

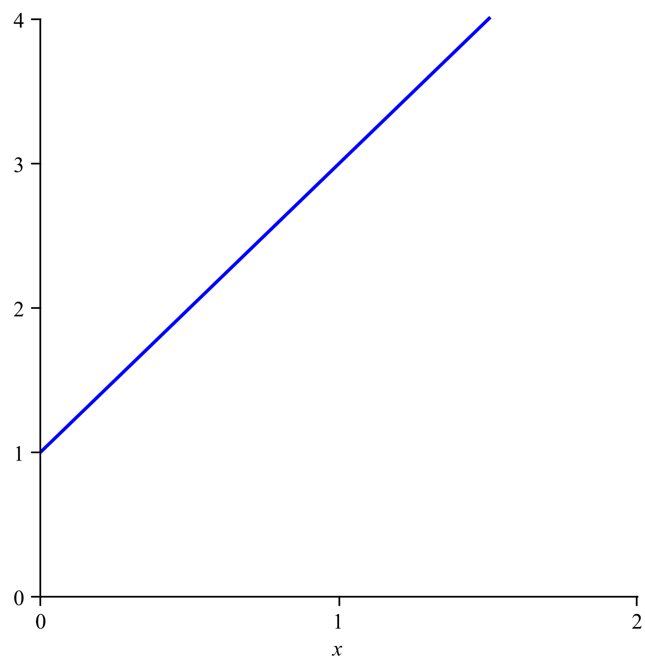
Limits

Recall the definition of a limit

$$\lim_{x \to a} f(x) = L \tag{11}$$

which simply means as x gets close to a, then $f(x)$ gets close to L. The following example illustrates

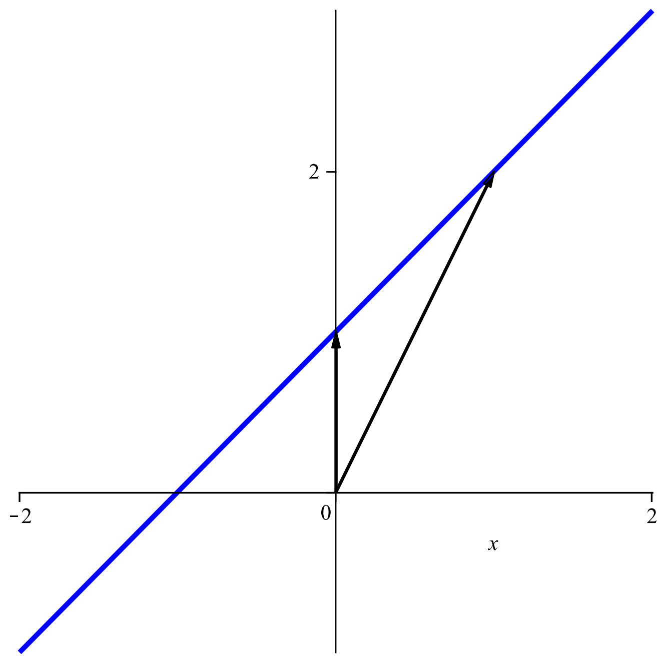
$$\lim_{x \to 1} 2x + 1 = 3 \tag{12}$$



So what about

$$\lim_{t \to 1} < t, t+1 > = ? \tag{13}$$

Well, it still means getting close but now the vector function is approaching a particular vector (here, it's $< 1, 2 >$).



So mathematically

$$\lim_{t\to 1} < t, t+1 > = \left\langle \lim_{t\to 1} t,\ \lim_{t\to 1} t+1 \right\rangle =< 1, 2 > . \tag{14}$$

In general

$$\lim_{t\to a} < f(t), g(t) >= \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t) \right\rangle \tag{15}$$

or

$$\lim_{t\to a} < f(t), g(t), h(t) >= \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) \right\rangle \tag{16}$$

depending on whether we have 2D or 3D vector functions. So all the rules that apply from Calc 1 now apply to vector functions.

Example 1

Find

$$\lim_{t\to 0} < \cos t, \sin t > . \tag{17}$$

In this example, it's a simple substitution

$$\lim_{t\to 0} < \cos t, \sin t > = < \cos 0, \sin 0 > = < 1, 0 > \tag{18}$$

Example 2

Find

$$\lim_{t\to 1} \left\langle \frac{t^2-1}{t-1}, \frac{2t}{t^2-1} \right\rangle. \tag{19}$$

The first limit is defined since

$$\lim_{t\to 1} \frac{t^2-1}{t-1} = \lim_{t\to 1} \frac{(t+1)(t-1)}{t-1} = 2 \tag{20}$$

However, the second limit is undefined since

$$\lim_{t\to 1} \frac{2t}{t^2-1} = \pm\infty \tag{21}$$

and so in this example

$$\lim_{t\to 1} \left\langle \frac{t^2-1}{t-1}, \frac{2t}{t^2-1} \right\rangle = DNE \tag{22}$$

Continuity

From Calculus 1, we define continuity as

$$\begin{aligned} (i) &\quad \lim_{x\to a} f(x) \text{ exists} \\ (ii) &\quad f(a) \text{ exists} \\ (iii) &\quad \lim_{x\to a} f(x) = f(a) \end{aligned} \tag{23}$$

As similar definition applies for vector functions

$$
\begin{aligned}
(i) &\quad \lim_{t\to a} \vec{r}(t) \;\text{ exists} \\
(ii) &\quad \vec{r}(a) \;\text{ exists} \\
(iii) &\quad \lim_{t\to a} \vec{r}(t) = \vec{r}(a)
\end{aligned}
\tag{24}
$$

Example 3

Is

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle \tag{25}$$

continuous at $t = 0$? We see

$$
\begin{aligned}
(i) &\quad \lim_{t\to 0} \vec{r}(t) =< 1, 0, 0 > \quad \text{exists} \\
(ii) &\quad \vec{r}(0) =< 1, 0, 0 > \quad \text{exists} \\
(iii) &\quad \lim_{t\to 0} \vec{r}(t) = \vec{r}(0) \quad \text{yes}
\end{aligned}
\tag{26}
$$

so the vector function is continuous at $t = 0$.