Math 4381/6378 Symmetry

Last class we obtained the symmetries of the PDE

$$u_t = u_x^2$$

which were

$$T = c_1 x^2 + (c_2 t + c_3)x + c_4 t^2 + c_9 t + c_{10}$$

$$X = -2(2c_1 x + c_2 t + c_3)u + \frac{c_2}{2}x^2$$

$$+ (c_4 t + c_5)x + c_6 t + c_7$$

$$U = -4c_1 u^2 + (c_2 x - c_9 + c_5)u - \frac{c_4}{4}x^2 - \frac{c_6}{2}x + c_8$$

So the question is - what do we do with them?

With ODEs we introduced new variables r & s

where $X r_x + Y r_y = 0 \quad X s_x + Y s_y = 1$

so that the new ODE is independent of s

Let's consider the generator

$$\Gamma = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}$$

Under a change of variables

$$f_x = f_r r_x + f_s s_x, \quad f_y = f_r r_y + f_s s_y$$

so $\Gamma = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y}$

$$= X\left(r_x \frac{\partial}{\partial r} + s_x \frac{\partial}{\partial s}\right) + Y\left(r_y \frac{\partial}{\partial r} + s_y \frac{\partial}{\partial s}\right)$$

$$= (X r_x + Y r_y) \frac{\partial}{\partial r} + (X s_x + Y s_y) \frac{\partial}{\partial s}$$

and choosing

$$X r_x + Y r_y = 0 \quad X s_x + Y s_y = 1$$

gives $\quad \Gamma = \frac{\partial}{\partial s}$

meaning new eq^n has no s!

Consider symmetry generator for PDEs

$$\Gamma = T \frac{\partial}{\partial t} + X \frac{\partial}{\partial x} + U \frac{\partial}{\partial u}$$

and a change of variables

$$(t, x, u) \to (r, s, v)$$

so $\Gamma = T\left(r_t \frac{\partial}{\partial r} + s_t \frac{\partial}{\partial s} + v_t \frac{\partial}{\partial v}\right)$

$\quad + X\left(r_x \frac{\partial}{\partial r} + s_x \frac{\partial}{\partial s} + v_x \frac{\partial}{\partial v}\right)$

$\quad + U\left(r_u \frac{\partial}{\partial r} + s_u \frac{\partial}{\partial u} + v_u \frac{\partial}{\partial v}\right)$

$$= (T r_t + X r_x + U r_u)\frac{\partial}{\partial r}$$

$$+ (T s_t + X s_x + U s_u)\frac{\partial}{\partial s}$$

$$+ (T v_t + X v_x + U v_u)\frac{\partial}{\partial u}$$

If we choose

$$T r_t + X r_x + U r_u = 0$$

$$T s_t + X s_x + U s_u = 1$$

$$T v_t + S v_x + U v_u = 0$$

then new generator is

$\Gamma = \frac{\partial}{\partial s}$ meaning new eq^n will have no s. let's look at an example ex! $c_9 = 0$ others $= 0$

$T = t,\ X = 0,\ U = -u$

so we solve
$$t r_t - u r_u = 0$$
$$t s_t - u s_u = 1$$
$$t v_t - u v_u = 0$$

M of C
$$\frac{dt}{t} = \frac{dx}{0} = \frac{du}{-u} = \frac{dr}{0}$$

$c_1 = tu,\ c_2 = x,\ c_3 = r$

$r = R(tu, x)$

similarly

$s = \ln t + \bar{S}(tu, x)$

$v = \bar{V}(tu, x)$

so pick $r = x,\ s = \ln t,\ v = tu$

or $t = e^s\ \ x = r\ \ u = \frac{v}{e^s}$

& $u_t = u_x^2$ becomes $-e^{-2s}(v - v_s - v_r^2) = 0$

or $v_s + v_r^2 - v = 0$ no s!

ex 2 $\quad c_4 = 1 \quad$ others $= 0$

$$T = t^2 \quad X = tx, \quad U = -\frac{x^2}{4}$$

we solve
$$t^2 r_t + tx r_x - \frac{x^2}{4} r_u = 0$$
$$t^2 s_t + tx s_x - \frac{x^2}{4} s_u = 1$$
$$t^2 v_t + tx v_x - \frac{x^2}{4} v_u = 0$$

which gives

$$r = R\left(\frac{x}{t}, u + \frac{x^2}{4t}\right), \quad s = -\frac{1}{t} + S\left(\frac{x}{t}, u + \frac{x^2}{4t}\right)$$
$$v = V\left(\frac{x}{t}, u + \frac{x^2}{4t}\right)$$

choosing $\quad r = \frac{x}{t}, \quad s = -\frac{1}{t}, \quad v = u + \frac{x^2}{4t}$

or $\quad t = -\frac{1}{s}, \quad x = -\frac{r}{s}, \quad u = v + \frac{r^2}{4s}$

transforms

$$u_t = u_x^2 \Rightarrow \quad -s^2\left(v_r^2 - v_s\right) = 0$$

or $\quad v_s - v_r^2 = 0 \quad$ no s!

<u>ex3</u> $c_3 = 1$ others $= 0$

$$T = x, \quad X = -2u \quad U = 0$$

we solve $\quad x r_t - 2u r_x = 0$

$$x s_t - 2u s_x = 1$$

$$x v_t - 2u v_x = 0$$

sol^n $\quad r = R(x^2 + 4tu, u), \quad s = -\dfrac{x}{2u} + S(x^2+4tu, u)$

$$v = V(x^2 + 4tu, u)$$

If we choose

$$r = x^2 + 4tu, \quad s = -\frac{x}{2u}, \quad v = u$$

or $\quad t = \dfrac{r - 4s^2v^2}{4v}, \quad x = -2sv, \quad u = v$

cl. $u_t = u_x^2$ becomes

$$v_s^2 + 16 r v^2 v_r^2 - 16 v^3 v_r = 0 \quad \text{no } s!$$

So why bother? We transformed the PDE

$u_t = u_x^2$ to $v_s + v_r^2 - v = 0$

$v_s - v_r^2 = 0$

$v_s^2 + 16 r v^2 v_r^3 - 16 v^3 v_r = 0$

Still more PDEs!

Now suppose that $v_s = 0$ (fcns no s)

then $v_s + v_r^2 - v = 0$ becom $v_r^2 - v = 0$ an ODE

$v_s^2 + 16 v^2 v_r (r v_r^2 - v) = 0$ becomes $r v_r^2 - v = 0$ ODE

1st Ex $v_r = \pm\sqrt{v}$ $\quad \dfrac{dv}{v^{1/2}} = \pm dr$

$2 v^{1/2} = \pm r + c_1$

$\Rightarrow \quad v = \left(\dfrac{\pm r + c_1}{2}\right)^2$

$t u = \dfrac{(\pm x + c_1)^2}{4}$ or $u = \dfrac{(x+c)^2}{4t}$ an exact sol^n

Can we bypass the introduction of the new variables r, s, v. — Yes!

Note: in the 1st example

$$t = e^{s}, \quad x = r, \quad u = \frac{v}{e^{s}}$$

Now $v = f(r)$ only so

$$u = \frac{f(r)}{e^{s}} = \frac{f(x)}{t}.$$

Now $u_t = -\frac{f(x)}{t^2}$,

& $t u_t + u = 0$ or $t u_t = -u$

Recall $T = t \quad X = 0 \quad \bar{U} = -u$

so this is $T u_t + X u_x = \bar{U}$

Called invariant surface condition

Invariant Surface Condition (ISC)

Consider the sol^n

$$u = f(t,x)$$

invariant under

$$\bar{t} = t + \varepsilon T, \quad \bar{x} = x + \varepsilon X, \quad \bar{u} = u + \varepsilon U$$

so $\quad f(\bar{t}, \bar{x}) = \bar{u}$

$$\Rightarrow \quad f(t + \varepsilon T, x + \varepsilon X) = u + \varepsilon U$$

$$f(t,x) + \varepsilon\left[T f_t + X f_x \right] = u + \varepsilon U$$

$\varepsilon)$ to $O(\varepsilon^2)$ $\quad$ if $\quad u = f(t,x)$

then $\quad T f_t + X f_x = U$

$\because$ $u = f$ then

$$T u_t + X u_x = U$$