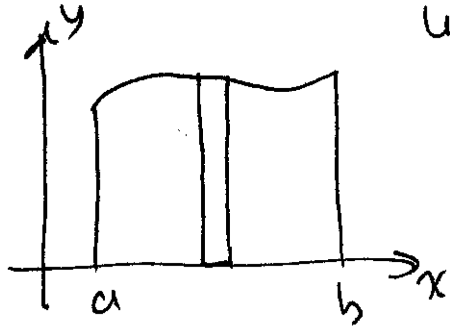
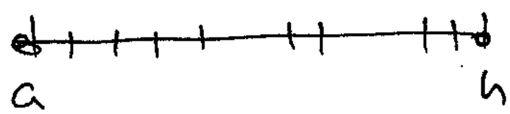
Math 1496 - Calc 1

Area Under Curve's

So far we have been approximating areas under curve with rectangles.
In general if we have $y = f(x)$ on $[a,b]$ and $f(x) \geq 0$, we first subdivide the interval into n equal pieces





so each piece has length and thus we call Δx so $\Delta x = \frac{b-a}{n}$ (width of each rectangle)

For the i^{th} rectangle, the right endpt is

$$x_i^* = a + \frac{b-a}{n} i$$

The height of this rectangle

$$h_i = f(x_i^*) = f\left(a + \frac{b-a}{n} i\right)$$

The area of the i^{th} rectangle

$$A_i = f(x_i^*)\Delta x$$

$$= f\left(a + \frac{b-a}{n} i\right)\frac{b-a}{n}$$

Add up the rectangles & let # rectangles $\to \infty$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a+\frac{b-a}{n} i\right)\frac{b-a}{n}$$

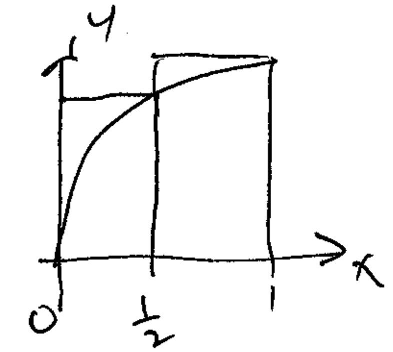
ex $f(x) = \sqrt{x}$ on $[0,1]$

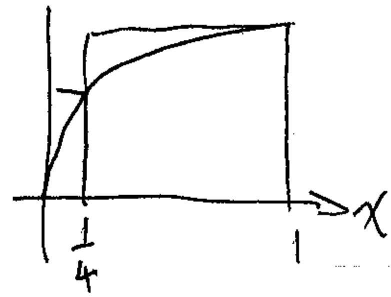
so $\Delta x = \frac{1}{n}$ $x_i = \frac{i}{n}$ $h_i = \sqrt{\frac{i}{n}}$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i}{n}}\,\frac{1}{n}$$

problem here - we don't know $\sum_{i=1}^{n} \sqrt{i}$

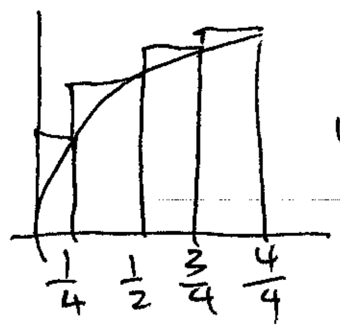
so instead of having say 2 rectangles of equal thickness, how about $\frac{1}{4}$ & $\frac{3}{4}$



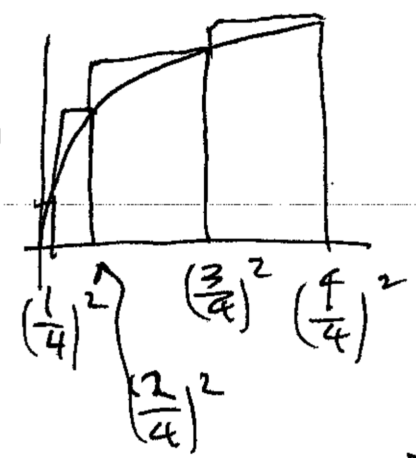


why? $\sqrt{\frac{1}{4}} = \frac{1}{2}$ (easy to calculate)

If 4 rect.



instead



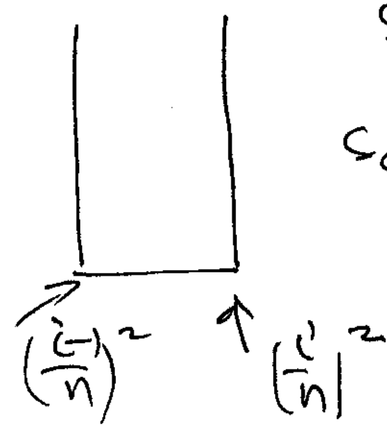
why? the heights are easier to calculate.

However the thickness' change

So in general if we have n rectangles

$$x_i^* = \left(\frac{i}{n}\right)^2 \quad \text{so} \quad h_i = f(x_i^*) = \frac{i}{n}$$

Rectangle thickness



So we subtract these 2

so $\Delta x_i = \left(\frac{i}{n}\right)^2 - \left(\frac{i-1}{n}\right)^2$

$$= \frac{i^2-(i-1)^2}{n^2}$$

$$= \frac{i^2-(i^2-2i+1)}{n^2}$$

$$= \frac{2i-1}{n^2}$$

Now

$$A_i = f(x_i^*)\,\Delta x_i$$

$$= \sqrt{\left(\frac{i}{n}\right)^2}\;\frac{2i-1}{n^2}$$

$$= \frac{i}{n}\cdot\frac{2i-1}{n^2}$$

<u>note</u>: we use

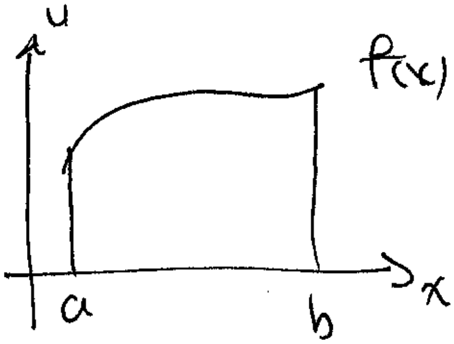
$\Delta x_i \leftarrow$ for ith rectangle

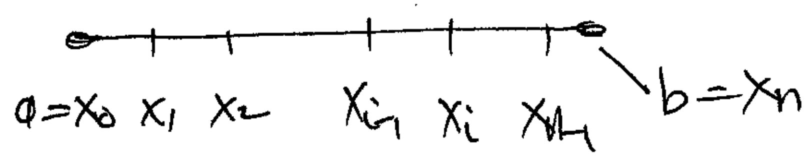
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i^2-i}{n^3} = \lim_{n\to\infty} \frac{2\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i}{n^3}$$

$$= \lim_{n\to\infty} \frac{2}{n^3}\,\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2n^3}$$

$$= \frac{4}{6} = \frac{2}{3}$$

In general, subdivide interval $[a,b]$ into n pieces





thickness of i^{th} rectangle

$$\Delta x_i = x_i - x_{i-1}$$

For the height of this rectangle, we were picking the right endpt but as long as we pick a pt inside $[x_{i-1}, x_i]$ we're ok

This we'll call $c_i \in [x_{i-1}, x_i]$

so $h_i = f(c_i)$

Area of i^{th} rectangle $A_i = f(c_i)\Delta x_i$

Now $A = \lim\limits_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(c_i)\Delta x_i$ ← Called Riemann Sum

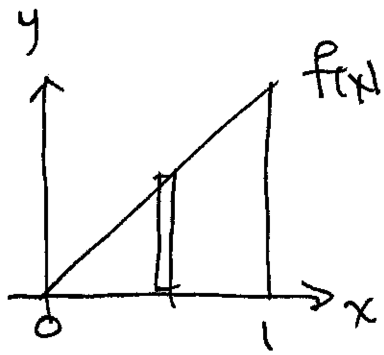
<u>Def$^{\underline{n}}$</u> <u>Definite Integral</u>

If f is defined on $[a,b]$ then

$$\lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx$$

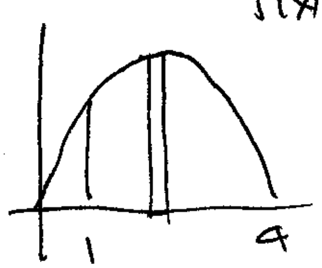
<u>Note</u>
$$\lim \sum \to \int$$
$$f(c_i) \longrightarrow f(x)$$
$$\Delta x_i \longrightarrow dx$$

so



$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)\frac{1}{n} = \int_0^1 x\,dx$$

$f(x) = 4x - x^2$



$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[4\left(1+\frac{3}{n}i\right) - \left(1+\frac{3i}{n}\right)^2\right]\frac{3}{n} = \int_1^4 (4x - x^2)\,dx$$