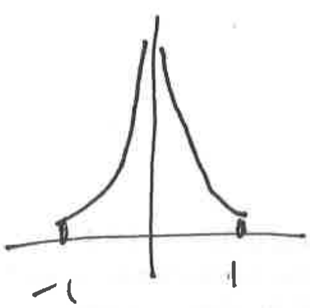
Math 1497 Calc 2

Consider
$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -\frac{1}{1} - \left(-\frac{1}{-1}\right) = -2$$

but $\frac{1}{x^2} \geq 0$ so $\int_{-1}^{1} \frac{dx}{x^2} \geq 0$ ← should be!

so what went wrong?

consider the picture



clearly $\frac{1}{x^2} \geq 0$

but $\frac{1}{x^2}$ is undefined at $x = 0$

Consider
$$\int_{0}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{0}^{1} = -\frac{1}{1} + \frac{1}{0} \leftarrow \text{undefined}$$

so we have to be careful integrating over a singularity!

Type 1 Improper Integral for $\int_a^b f(x)dx$

if a singularity is at $x=b$

$$\lim_{M\to b}\int_a^M f(x)dx$$ if limit exists $\int$ "converges"; if limit DNE $\int$ "diverges"

if at $x=a$

$$\lim_{m\to a}\int_m^b f(x)dx$$

and if at c where $a<c<b$

$$\int_a^c f(x)dx + \int_c^b f(x)dx$$

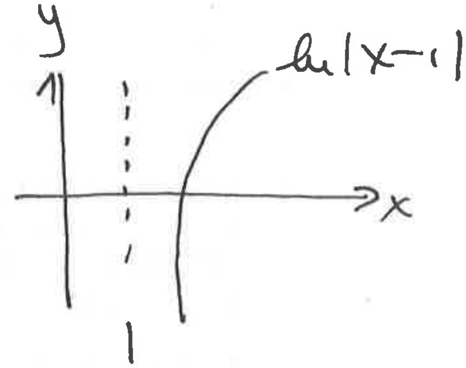
1st case 2nd case

if either diverges, the entire $\int$ diverges

Ex 1 $\displaystyle\int_1^2 \frac{dx}{x-1} = \lim_{a\to 1} \int_a^2 \frac{dx}{x-1}$

$$= \lim_{a\to 1} \ln|x-1|\Big|_a^2 = \lim_{a\to 1} \underbrace{\ln|\cancel{2}-1|}_{0} - \ln|a-1|$$

$$= -\lim_{a\to 1} \ln|a-1| = \infty$$



So $\int$ diverges

ex 2 $\displaystyle\int_1^4 \frac{dx}{\sqrt{4-x}}$ singularity at $x=4$

$$\lim_{b\to 4} \int_1^b \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4} -2\sqrt{4-x}\Big|_1^b$$

$$= -2 \lim_{b\to 4} \sqrt{4-b} - \sqrt{4-1} = -2\{0-\sqrt{3}\} = 2\sqrt{3}$$

So here $\int$ converges

Ex 3 $\displaystyle\int_{-1}^{1} \frac{dx}{x^2-1}$ singular at both endpts

$$\int_{-1}^{1} \frac{1}{2}\frac{1}{x-1} - \frac{1}{2(x+1)}\, dx$$

$$\frac{1}{2}\int_{-1}^{1}\frac{dx}{x-1} - \frac{1}{2}\int_{-1}^{1}\frac{dx}{x+1}$$

$$\frac{1}{2}\lim_{b\to 1}\int_{-1}^{b}\frac{dx}{x-1} - \frac{1}{2}\lim_{a\to -1}\int_{a}^{1}\frac{dx}{x+1}$$

diverges so entire $\int$ diverge

Ex 4 $\displaystyle\int_{-1}^{1}\frac{dx}{x^2} = \int_{-1}^{0}\frac{dx}{x^2} + \int_{0}^{1}\frac{dx}{x^2}$

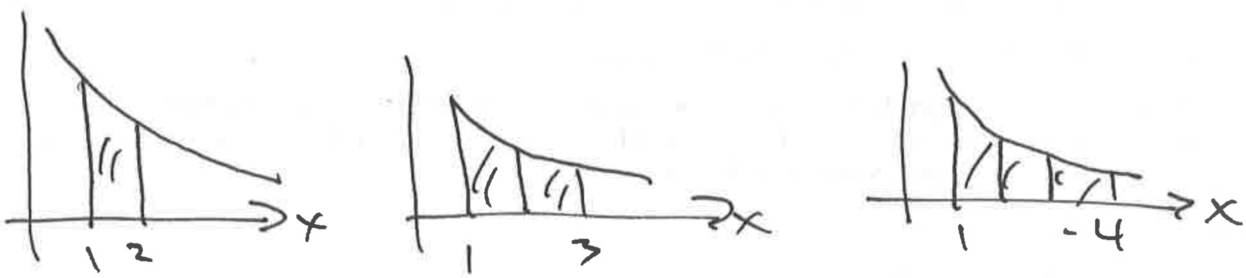
$$\lim_{a\to 0}\int_{a}^{1}\frac{dx}{x^2} = \lim_{a\to 0} -\frac{1}{x}\Big|_{a}^{1} = \lim_{a\to 0} -1 + \frac{1}{a} \to \infty$$

so entire $\int$ diverge

Consider $\int_1^2 \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^2 = -\frac{1}{2}+1 = \frac{1}{2}$

$$\int_1^3 \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^3 = -\frac{1}{3}+1 = \frac{2}{3}$$

$$\int_1^4 \frac{dx}{x^2} = \frac{3}{4} \text{ etc}$$



what about $\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$

$$= \lim_{b\to\infty} -\frac{1}{b}+1 \to 1$$

so $\int$ converges

Infinite Limits

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

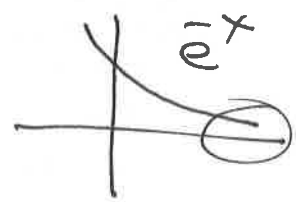
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

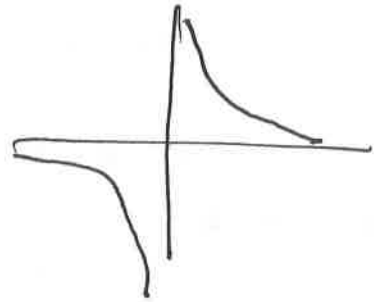
check either or both

Ex $$\int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b$$

$$= \lim_{b\to\infty} -e^{-b} + e^{-1} - e^{-1}$$



ex $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x}$



$$= \lim_{a\to-\infty} \int_a^{-1} \frac{dx}{x}$$

$$= \lim_{a\to-\infty} \ln|x| \Big|_a^{-1} = \lim_{a\to-\infty} \ln|-1| - \ln|a| \to \infty$$

So $\int$ diverges

ex $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \tan^{-1}x \Big|_a^0 + \lim_{b\to\infty} \tan^{-1}x \Big|_0^b$$

$$= \lim_{a\to-\infty} \tan^{-1}0 - \tan^{-1}a + \lim_{b\to\infty} \tan^{-1}b - \tan^{-1}0$$

$$= -(-\pi/2) + \pi/2 = \pi$$

So $\int$ conv.