

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 1

#### Question:

Find  $\frac{dy}{dx}$  for each of the following, leaving your answer in terms of the parameter  $t$ :

(a)  $x = 2t, y = t^2 - 3t + 2$

(b)  $x = 3t^2, y = 2t^3$

(c)  $x = t + 3t^2, y = 4t$

(d)  $x = t^2 - 2, y = 3t^5$

(e)  $x = \frac{2}{t}, y = 3t^2 - 2$

(f)  $x = \frac{1}{2t-1}, y = \frac{t^2}{2t-1}$

(g)  $x = \frac{2t}{1+t^2}, y = \frac{1-t^2}{1+t^2}$

(h)  $x = t^2 e^t, y = 2t$

(i)  $x = 4 \sin 3t, y = 3 \cos 3t$

(j)  $x = 2 + \sin t, y = 3 - 4 \cos t$

(k)  $x = \sec t, y = \tan t$

(l)  $x = 2t - \sin 2t, y = 1 - \cos 2t$

#### Solution:

(a)  $x = 2t, y = t^2 - 3t + 2$   
 $\frac{dx}{dt} = 2, \frac{dy}{dt} = 2t - 3$

Using the chain rule

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2t-3}{2}$$

(b)  $x = 3t^2, y = 2t^3$

$$\frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2$$

Using the chain rule

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{6t^2}{6t} = t$$

(c)  $x = t + 3t^2, y = 4t$

$$\frac{dx}{dt} = 1 + 6t, \frac{dy}{dt} = 4$$

$$\therefore \frac{dy}{dx} = \frac{4}{1+6t} \quad (\text{from the chain rule})$$

(d)  $x = t^2 - 2, y = 3t^5$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 15t^4$$

$$\therefore \frac{dy}{dx} = \frac{15t^4}{2t} = \frac{15t^3}{2} \quad (\text{from the chain rule})$$

(e)  $x = \frac{2}{t}, y = 3t^2 - 2$

$$\frac{dx}{dt} = -2t^{-2}, \frac{dy}{dt} = 6t$$

$$\therefore \frac{dy}{dx} = \frac{6t}{-2t^{-2}} = -3t^3 \quad (\text{from the chain rule})$$

(f)  $x = \frac{1}{2t-1}, y = \frac{t^2}{2t-1}$

$$\text{As } x = (2t-1)^{-1}, \frac{dx}{dt} = -2(2t-1)^{-2} \quad (\text{from the chain rule})$$

Use the quotient rule to give

$$\frac{dy}{dt} = \frac{(2t-1)(2t) - t^2(2)}{(2t-1)^2} = \frac{2t^2 - 2t}{(2t-1)^2} = \frac{2t(t-1)}{(2t-1)^2}$$

$$\text{Hence } \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{2t(t-1)}{(2t-1)^2} \div -2(2t-1)^{-2}$$

$$= \frac{2t(t-1)}{(2t-1)^2} \div \frac{-2}{(2t-1)^2}$$

$$= \frac{2t(t-1)}{(2t-1)^2} \times \frac{(2t-1)^2}{-2}$$

$$= -t(t-1) \text{ or } t(1-t)$$

$$\text{(g) } x = \frac{2t}{1+t^2}, y = \frac{1-t^2}{1+t^2}$$

$$\frac{dx}{dt} = \frac{(1+t^2)2 - 2t(2t)}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2}$$

and

$$\frac{dy}{dt} = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} = \frac{-4t}{(1+t^2)^2}$$

Hence

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{-4t}{(1+t^2)^2} \div \frac{2-2t^2}{(1+t^2)^2}$$

$$= \frac{-4t}{2(1-t^2)}$$

$$= -\frac{2t}{(1-t^2)} \text{ or } \frac{2t}{t^2-1}$$

$$\text{(h) } x = t^2e^t, y = 2t$$

$$\frac{dx}{dt} = t^2e^t + e^t2t \text{ (from the product rule) and } \frac{dy}{dt} = 2$$

$$\therefore \frac{dy}{dx} = \frac{2}{t^2 e^t + 2te^t} = \frac{2}{te^t(t+2)} \quad (\text{from the chain rule})$$

(i)  $x = 4 \sin 3t, y = 3 \cos 3t$

$$\frac{dx}{dt} = 12 \cos 3t, \frac{dy}{dt} = -9 \sin 3t$$

$$\therefore \frac{dy}{dx} = \frac{-9 \sin 3t}{12 \cos 3t} = -\frac{3}{4} \tan 3t \quad (\text{from the chain rule})$$

(j)  $x = 2 + \sin t, y = 3 - 4 \cos t$

$$\frac{dx}{dt} = \cos t, \frac{dy}{dt} = 4 \sin t$$

$$\therefore \frac{dy}{dx} = \frac{4 \sin t}{\cos t} = 4 \tan t \quad (\text{from the chain rule})$$

(k)  $x = \sec t, y = \tan t$

$$\frac{dx}{dt} = \sec t \tan t, \frac{dy}{dt} = \sec^2 t$$

$$\text{Hence } \frac{dy}{dx} = \frac{\sec^2 t}{\sec t \tan t}$$

$$= \frac{\sec t}{\tan t}$$

$$= \frac{1}{\cos t} \times \frac{\cos t}{\sin t}$$

$$= \frac{1}{\sin t}$$

$$= \operatorname{cosec} t$$

(l)  $x = 2t - \sin 2t, y = 1 - \cos 2t$

$$\frac{dx}{dt} = 2 - 2 \cos 2t, \frac{dy}{dt} = 2 \sin 2t$$

$$\text{Hence } \frac{dy}{dx} = \frac{2 \sin 2t}{2 - 2 \cos 2t}$$

$$= \frac{2 \times 2 \sin t \cos t}{2 - 2(1 - 2 \sin^2 t)} \quad (\text{using double angle formulae})$$

$$= \frac{\sin t \cos t}{\sin^2 t}$$

$$= \frac{\cos t}{\sin t}$$

$$= \cot t$$

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# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 2

#### Question:

(a) Find the equation of the tangent to the curve with parametric equations

$$x = 3t - 2 \sin t, y = t^2 + t \cos t, \text{ at the point } P, \text{ where } t = \frac{\pi}{2}.$$

(b) Find the equation of the tangent to the curve with parametric equations

$$x = 9 - t^2, y = t^2 + 6t, \text{ at the point } P, \text{ where } t = 2.$$

#### Solution:

(a)  $x = 3t - 2 \sin t, y = t^2 + t \cos t$

$$\frac{dx}{dt} = 3 - 2 \cos t, \frac{dy}{dt} = 2t + \left( -t \sin t + \cos t \right)$$

$$\therefore \frac{dy}{dx} = \frac{2t - t \sin t + \cos t}{3 - 2 \cos t}$$

$$\text{When } t = \frac{\pi}{2}, \frac{dy}{dx} = \frac{\left( \pi - \frac{\pi}{2} \right)}{3} = \frac{\pi}{6}$$

$\therefore$  the tangent has gradient  $\frac{\pi}{6}$ .

$$\text{When } t = \frac{\pi}{2}, x = \frac{3\pi}{2} - 2 \text{ and } y = \frac{\pi^2}{4}$$

$\therefore$  the tangent passes through the point  $\left( \frac{3\pi}{2} - 2, \frac{\pi^2}{4} \right)$

The equation of the tangent is

$$y - \frac{\pi^2}{4} = \frac{\pi}{6} \left[ x - \left( \frac{3\pi}{2} - 2 \right) \right]$$

$$\therefore y - \frac{\pi^2}{4} = \frac{\pi}{6}x - \frac{\pi^2}{4} + \frac{\pi}{3}$$

$$\text{i.e. } y = \frac{\pi}{6}x + \frac{\pi}{3}$$

(b)  $x = 9 - t^2, y = t^2 + 6t$

$$\frac{dx}{dt} = -2t, \quad \frac{dy}{dt} = 2t + 6$$

$$\therefore \frac{dy}{dx} = \frac{2t + 6}{-2t}$$

$$\text{At the point where } t = 2, \quad \frac{dy}{dx} = \frac{10}{-4} = \frac{-5}{2}$$

Also at  $t = 2$ ,  $x = 5$  and  $y = 16$ .

$\therefore$  the tangent has equation

$$y - 16 = \frac{-5}{2} (x - 5)$$

$$\therefore 2y - 32 = -5x + 25$$

$$\text{i.e. } 2y + 5x = 57$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 3

#### Question:

- (a) Find the equation of the normal to the curve with parametric equations  $x = e^t$ ,  $y = e^t + e^{-t}$ , at the point  $P$ , where  $t = 0$ .
- (b) Find the equation of the normal to the curve with parametric equations  $x = 1 - \cos 2t$ ,  $y = \sin 2t$ , at the point  $P$ , where  $t = \frac{\pi}{6}$ .

#### Solution:

(a)  $x = e^t$ ,  $y = e^t + e^{-t}$

$$\frac{dx}{dt} = e^t \text{ and } \frac{dy}{dt} = e^t - e^{-t}$$

$$\therefore \frac{dy}{dx} = \frac{e^t - e^{-t}}{e^t}$$

When  $t = 0$ ,  $\frac{dy}{dx} = 0$

- $\therefore$  gradient of curve is 0  
 $\therefore$  normal is parallel to the  $y$ -axis.

When  $t = 0$ ,  $x = 1$  and  $y = 2$

- $\therefore$  equation of the normal is  $x = 1$

(b)  $x = 1 - \cos 2t$ ,  $y = \sin 2t$

$$\frac{dx}{dt} = 2 \sin 2t \text{ and } \frac{dy}{dt} = 2 \cos 2t$$

$$\therefore \frac{dy}{dx} = \frac{2 \cos 2t}{2 \sin 2t} = \cot 2t$$

When  $t = \frac{\pi}{6}$ ,  $\frac{dy}{dx} = \frac{1}{\tan \frac{\pi}{3}} = \frac{1}{\sqrt{3}}$

- $\therefore$  gradient of the normal is  $-\sqrt{3}$

When  $t = \frac{\pi}{6}$ ,  $x = 1 - \cos \frac{\pi}{3} = \frac{1}{2}$  and  $y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$



∴ equation of the normal is

$$y - \frac{\sqrt{3}}{2} = -\sqrt{3} \left( x - \frac{1}{2} \right)$$

i.e.  $y - \frac{\sqrt{3}}{2} = -\sqrt{3}x + \frac{\sqrt{3}}{2}$

$$\therefore y + \sqrt{3}x = \sqrt{3}$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 4

#### Question:

Find the points of zero gradient on the curve with parametric equations  $x = \frac{t}{1-t}$ ,  $y = \frac{t^2}{1-t}$ ,  $t \neq 1$ .

You do not need to establish whether they are maximum or minimum points.

#### Solution:

$$x = \frac{t}{1-t}, y = \frac{t^2}{1-t}$$

Use the quotient rule to give

$$\frac{dx}{dt} = \frac{(1-t) \times 1 - t(-1)}{(1-t)^2} = \frac{1}{(1-t)^2}$$

and

$$\frac{dy}{dt} = \frac{(1-t)2t - t^2(-1)}{(1-t)^2} = \frac{2t - t^2}{(1-t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2t - t^2}{(1-t)^2} \div \frac{1}{(1-t)^2} = t(2 - t)$$

When  $\frac{dy}{dx} = 0$ ,  $t = 0$  or  $2$

When  $t = 0$  then  $x = 0$ ,  $y = 0$

When  $t = 2$  then  $x = -2$ ,  $y = -4$

$\therefore (0, 0)$  and  $(-2, -4)$  are the points of zero gradient.

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise B, Question 1

#### Question:

Find an expression in terms of  $x$  and  $y$  for  $\frac{dy}{dx}$ , given that:

(a)  $x^2 + y^3 = 2$

(b)  $x^2 + 5y^2 = 14$

(c)  $x^2 + 6x - 8y + 5y^2 = 13$

(d)  $y^3 + 3x^2y - 4x = 0$

(e)  $3y^2 - 2y + 2xy = x^3$

(f)  $x = \frac{2y}{x^2 - y}$

(g)  $(x - y)^4 = x + y + 5$

(h)  $e^xy = xe^y$

(i)  $\sqrt{xy} + x + y^2 = 0$

#### Solution:

(a)  $x^2 + y^3 = 2$

Differentiate with respect to  $x$ :

$$2x + 3y^2 \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{3y^2}$$

(b)  $x^2 + 5y^2 = 14$

$$2x + 10y \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{10y} = -\frac{x}{5y}$$

$$(c) x^2 + 6x - 8y + 5y^2 = 13$$

$$2x + 6 - 8 \frac{dy}{dx} + 10y \frac{dy}{dx} = 0$$

$$2x + 6 = \left( 8 - 10y \right) \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2x + 6}{8 - 10y} = \frac{x + 3}{4 - 5y}$$

$$(d) y^3 + 3x^2y - 4x = 0$$

Differentiate with respect to  $x$ :

$$3y^2 \frac{dy}{dx} + \left( 3x^2 \frac{dy}{dx} + y \times 6x \right) - 4 = 0$$

$$\frac{dy}{dx} \left( 3y^2 + 3x^2 \right) = 4 - 6xy$$

$$\therefore \frac{dy}{dx} = \frac{4 - 6xy}{3(x^2 + y^2)}$$

$$(e) 3y^2 - 2y + 2xy - x^3 = 0$$

$$6y \frac{dy}{dx} - 2 \frac{dy}{dx} + \left( 2x \frac{dy}{dx} + y \times 2 \right) - 3x^2 = 0$$

$$\frac{dy}{dx} \left( 6y - 2 + 2x \right) = 3x^2 - 2y$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - 2y}{2x + 6y - 2}$$

$$(f) x = \frac{2y}{x^2 - y}$$

$$\therefore x^3 - xy = 2y$$

$$\text{i.e. } x^3 - xy - 2y = 0$$

Differentiate with respect to  $x$ :

$$3x^2 - \left( x \frac{dy}{dx} + y \times 1 \right) - 2 \frac{dy}{dx} = 0$$

$$3x^2 - y = \frac{dy}{dx} \left( x + 2 \right)$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - y}{x + 2}$$

$$(g) (x - y)^4 = x + y + 5$$

Differentiate with respect to  $x$ :

$$4(x - y)^3 \left( 1 - \frac{dy}{dx} \right) = 1 + \frac{dy}{dx} \text{ (The chain rule was used to}$$

differentiate the first term.)

$$\therefore 4(x - y)^3 - 1 = \frac{dy}{dx} \left[ 1 + 4(x - y)^3 \right]$$

$$\therefore \frac{dy}{dx} = \frac{4(x - y)^3 - 1}{1 + 4(x - y)^3}$$

$$(h) e^x y = x e^y$$

Differentiate with respect to  $x$ :

$$e^x \frac{dy}{dx} + y e^x = x e^y \frac{dy}{dx} + e^y \times 1$$

$$e^x \frac{dy}{dx} - x e^y \frac{dy}{dx} = e^y - y e^x$$

$$\frac{dy}{dx} \left( e^x - x e^y \right) = e^y - y e^x$$

$$\therefore \frac{dy}{dx} = \frac{e^y - y e^x}{e^x - x e^y}$$

$$(i) \sqrt{xy} + x + y^2 = 0$$

Differentiate with respect to  $x$ :

$$\frac{1}{2} (xy)^{-\frac{1}{2}} \left( x \frac{dy}{dx} + y \times 1 \right) + 1 + 2y \frac{dy}{dx} = 0$$

Multiply both sides by  $2\sqrt{xy}$ :

$$\left( x \frac{dy}{dx} + y \right) + 2\sqrt{xy} + 4y\sqrt{xy} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} \left( x + 4y\sqrt{xy} \right) = - \left( 2\sqrt{xy} + y \right)$$

$$\therefore \frac{dy}{dx} = \frac{- (2\sqrt{xy} + y)}{x + 4y\sqrt{xy}}.$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise B, Question 2

#### Question:

Find the equation of the tangent to the curve with implicit equation  $x^2 + 3xy^2 - y^3 = 9$  at the point (2, 1).

#### Solution:

$$x^2 + 3xy^2 - y^3 = 9$$

Differentiate with respect to  $x$ :

$$2x + \left[ 3x \left( 2y \frac{dy}{dx} \right) + y^2 \times 3 \right] - 3y^2 \frac{dy}{dx} = 0$$

When  $x = 2$  and  $y = 1$

$$4 + \left( 12 \frac{dy}{dx} + 3 \right) - 3 \frac{dy}{dx} = 0$$

$$\therefore 9 \frac{dy}{dx} = -7$$

$$\text{i.e. } \frac{dy}{dx} = \frac{-7}{9}$$

$\therefore$  the gradient of the tangent at (2, 1) is  $\frac{-7}{9}$ .

The equation of the tangent is

$$\left( y - 1 \right) = \frac{-7}{9} \left( x - 2 \right)$$

$$\therefore 9y - 9 = -7x + 14$$

$$\therefore 9y + 7x = 23$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise B, Question 3

### Question:

Find the equation of the normal to the curve with implicit equation  
 $(x + y)^3 = x^2 + y$  at the point  $(1, 0)$ .

### Solution:

$$(x + y)^3 = x^2 + y$$

Differentiate with respect to  $x$ :

$$3(x + y)^2 \left( 1 + \frac{dy}{dx} \right) = 2x + \frac{dy}{dx}$$

At the point  $(1, 0)$ ,  $x = 1$  and  $y = 0$

$$\therefore 3 \left( 1 + \frac{dy}{dx} \right) = 2 + \frac{dy}{dx}$$

$$\therefore 2 \frac{dy}{dx} = -1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-1}{2}$$

$\therefore$  The gradient of the normal at  $(1, 0)$  is 2.

$\therefore$  the equation of the normal is

$$y - 0 = 2(x - 1)$$

$$\text{i.e. } y = 2x - 2$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise B, Question 4

#### Question:

Find the coordinates of the points of zero gradient on the curve with implicit equation  $x^2 + 4y^2 - 6x - 16y + 21 = 0$ .

#### Solution:

$$x^2 + 4y^2 - 6x - 16y + 21 = 0 \quad \textcircled{1}$$

Differentiate with respect to  $x$ :

$$2x + 8y \frac{dy}{dx} - 6 - 16 \frac{dy}{dx} = 0$$

$$8y \frac{dy}{dx} - 16 \frac{dy}{dx} = 6 - 2x$$

$$\left( 8y - 16 \right) \frac{dy}{dx} = 6 - 2x$$

$$\therefore \frac{dy}{dx} = \frac{6 - 2x}{8y - 16}$$

$$\text{For zero gradient } \frac{dy}{dx} = 0 \Rightarrow 6 - 2x = 0 \Rightarrow x = 3$$

Substitute  $x = 3$  into  $\textcircled{1}$  to give

$$9 + 4y^2 - 18 - 16y + 21 = 0$$

$$\Rightarrow 4y^2 - 16y + 12 = 0 \quad [ \div 4 ]$$

$$\Rightarrow y^2 - 4y + 3 = 0$$

$$\Rightarrow (y - 1)(y - 3) = 0$$

$$\Rightarrow y = 1 \text{ or } 3$$

$\therefore$  the coordinates of the points of zero gradient are  $(3, 1)$  and  $(3, 3)$ .



# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise C, Question 1

#### Question:

Find  $\frac{dy}{dx}$  for each of the following:

(a)  $y = 3^x$

(b)  $y = \left(\frac{1}{2}\right)^x$

(c)  $y = xa^x$

(d)  $y = \frac{2^x}{x}$

#### Solution:

(a)  $y = 3^x$   
 $\frac{dy}{dx} = 3^x \ln 3$

(b)  $y = \left(\frac{1}{2}\right)^x$   
 $\frac{dy}{dx} = \left(\frac{1}{2}\right)^x \ln \frac{1}{2}$

(c)  $y = xa^x$   
 Use the product rule to give  
 $\frac{dy}{dx} = xa^x \ln a + a^x \times 1 = a^x \left( x \ln a + 1 \right)$

(d)  $y = \frac{2^x}{x}$   
 Use the quotient rule to give  
 $\frac{dy}{dx} = \frac{x \times 2^x \ln 2 - 2^x \times 1}{x^2} = \frac{2^x (x \ln 2 - 1)}{x^2}$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise C, Question 2

### Question:

Find the equation of the tangent to the curve  $y = 2^x + 2^{-x}$  at the point  $\left( 2, 4\frac{1}{4} \right)$ .

### Solution:

$$y = 2^x + 2^{-x}$$

$$\frac{dy}{dx} = 2^x \ln 2 - 2^{-x} \ln 2$$

$$\text{When } x = 2, \frac{dy}{dx} = 4 \ln 2 - \frac{1}{4} \ln 2 = \frac{15}{4} \ln 2$$

$\therefore$  the equation of the tangent at  $\left( 2, 4\frac{1}{4} \right)$  is

$$y - 4\frac{1}{4} = \frac{15}{4} \ln 2 \left( x - 2 \right)$$

$$\therefore 4y = (15 \ln 2) x + (17 - 30 \ln 2)$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise C, Question 3

### Question:

A particular radioactive isotope has an activity  $R$  millicuries at time  $t$  days given by the equation  $R = 200 (0.9)^t$ . Find the value of  $\frac{dR}{dt}$ , when  $t = 8$ .

### Solution:

$$R = 200 (0.9)^t$$

$$\frac{dR}{dt} = 200 \times \ln 0.9 \times (0.9)^t$$

Substitute  $t = 8$  to give

$$\frac{dR}{dt} = -9.07 \text{ (3 s.f.)}$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise C, Question 4

#### Question:

The population of Cambridge was 37 000 in 1900 and was about 109 000 in 2000. Find an equation of the form  $P = P_0 k^t$  to model this data, where  $t$  is measured as years since 1900. Evaluate  $\frac{dP}{dt}$  in the year 2000. What does this value represent?

#### Solution:

$$P = P_0 k^t$$

When  $t = 0$ ,  $P = 37\,000$

$$\therefore 37\,000 = P_0 \times k^0 = P_0 \times 1$$

$$\therefore P_0 = 37\,000$$

$$\therefore P = 37\,000 (k)^t$$

When  $t = 100$ ,  $P = 109\,000$

$$\therefore 109\,000 = 37\,000 (k)^{100}$$

$$\therefore k^{100} = \frac{109\,000}{37\,000}$$

$$\therefore k = 100\sqrt[100]{\frac{109}{37}} \approx 1.01$$

$$\frac{dP}{dt} = 37\,000 k^t \ln k$$

When  $t = 100$

$$\frac{dP}{dt} = 37\,000 \times \left( \frac{109}{37} \right) \times \ln k = 1000 \times 109 \times \frac{1}{100} \ln \frac{109}{37}$$

$$= 1178 \text{ people per year}$$

Rate of increase of the population during the year 2000.

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise D, Question 1

**Question:**

Given that  $V = \frac{1}{3}\pi r^3$  and that  $\frac{dV}{dt} = 8$ , find  $\frac{dr}{dt}$  when  $r = 3$ .

**Solution:**

$$V = \frac{1}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = \pi r^2$$

Using the chain rule

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\therefore 8 = \pi r^2 \times \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{8}{\pi r^2}$$

When  $r = 3$ ,  $\frac{dr}{dt} = \frac{8}{9\pi}$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise D, Question 2

**Question:**

Given that  $A = \frac{1}{4}\pi r^2$  and that  $\frac{dr}{dt} = 6$ , find  $\frac{dA}{dt}$  when  $r = 2$ .

**Solution:**

$$A = \frac{1}{4}\pi r^2$$

$$\frac{dA}{dr} = \frac{1}{2}\pi r$$

Using the chain rule

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = \frac{1}{2}\pi r \times 6 = 3\pi r$$

When  $r = 2$ ,  $\frac{dA}{dt} = 6\pi$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise D, Question 3

### Question:

Given that  $y = xe^x$  and that  $\frac{dx}{dt} = 5$ , find  $\frac{dy}{dt}$  when  $x = 2$ .

### Solution:

$$y = xe^x$$

$$\frac{dy}{dx} = xe^x + e^x \times 1$$

Using the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = e^x \left( x + 1 \right) \times 5$$

$$\text{When } x = 2, \frac{dy}{dt} = 15e^2$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise D, Question 4

### Question:

Given that  $r = 1 + 3 \cos \theta$  and that  $\frac{d\theta}{dt} = 3$ , find  $\frac{dr}{dt}$  when  $\theta = \frac{\pi}{6}$ .

### Solution:

$$r = 1 + 3 \cos \theta$$

$$\frac{dr}{d\theta} = -3 \sin \theta$$

Using the chain rule

$$\frac{dr}{dt} = \frac{dr}{d\theta} \times \frac{d\theta}{dt} = -3 \sin \theta \times 3 = -9 \sin \theta$$

$$\text{When } \theta = \frac{\pi}{6}, \frac{dr}{dt} = \frac{-9}{2}$$



# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise E, Question 1

#### Question:

In a study of the water loss of picked leaves the mass  $M$  grams of a single leaf was measured at times  $t$  days after the leaf was picked. It was found that the rate of loss of mass was proportional to the mass  $M$  of the leaf.

Write down a differential equation for the rate of change of mass of the leaf.

#### Solution:

$\frac{dM}{dt}$  represents rate of change of mass.

$\therefore \frac{dM}{dt} \propto -M$ , as rate of *loss* indicates a negative quantity.

$\therefore \frac{dM}{dt} = -kM$ , where  $k$  is the positive constant of proportionality.

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise E, Question 2

### Question:

A curve  $C$  has equation  $y = f(x)$ ,  $y > 0$ . At any point  $P$  on the curve, the gradient of  $C$  is proportional to the product of the  $x$  and the  $y$  coordinates of  $P$ .  
The point  $A$  with coordinates  $(4, 2)$  is on  $C$  and the gradient of  $C$  at  $A$  is  $\frac{1}{2}$ .

Show that  $\frac{dy}{dx} = \frac{xy}{16}$ .

### Solution:

The gradient of the curve is given by  $\frac{dy}{dx}$ .

$$\therefore \frac{dy}{dx} \propto xy \quad (\text{which is the product of } x \text{ and } y)$$

$$\therefore \frac{dy}{dx} = kxy, \text{ where } k \text{ is a constant of proportion.}$$

When  $x = 4$ ,  $y = 2$  and  $\frac{dy}{dx} = \frac{1}{2}$

$$\therefore \frac{1}{2} = k \times 4 \times 2$$

$$\therefore k = \frac{1}{16}$$

$$\therefore \frac{dy}{dx} = \frac{xy}{16}$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise E, Question 3

### Question:

Liquid is pouring into a container at a constant rate of  $30 \text{ cm}^3 \text{ s}^{-1}$ . At time  $t$  seconds liquid is leaking from the container at a rate of  $\frac{2}{15}V \text{ cm}^3 \text{ s}^{-1}$ , where  $V \text{ cm}^3$  is the volume of liquid in the container at that time.

Show that  $-15 \frac{dV}{dt} = 2V - 450$

### Solution:

Let the rate of increase of the volume of liquid be  $\frac{dV}{dt}$ .

$$\text{Then } \frac{dV}{dt} = 30 - \frac{2}{15}V$$

Multiply both sides by  $-15$ :

$$-15 \frac{dV}{dt} = 2V - 450$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise E, Question 4

### Question:

An electrically charged body loses its charge  $Q$  coulombs at a rate, measured in coulombs per second, proportional to the charge  $Q$ .

Write down a differential equation in terms of  $Q$  and  $t$  where  $t$  is the time in seconds since the body started to lose its charge.

### Solution:

The rate of change of the charge is  $\frac{dQ}{dt}$ .

$\therefore \frac{dQ}{dt} \propto -Q$ , as the body is *losing* charge the negative sign is required.

$\therefore \frac{dQ}{dt} = -kQ$ , where  $k$  is the positive constant of proportion.

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise E, Question 5

### Question:

The ice on a pond has a thickness  $x$  mm at a time  $t$  hours after the start of freezing. The rate of increase of  $x$  is inversely proportional to the square of  $x$ . Write down a differential equation in terms of  $x$  and  $t$ .

### Solution:

The rate of increase of  $x$  is  $\frac{dx}{dt}$ .

$\therefore \frac{dx}{dt} \propto \frac{1}{x^2}$ , as there is an *inverse* proportion.

$\therefore \frac{dx}{dt} = \frac{k}{x^2}$ , where  $k$  is the constant of proportion.

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise E, Question 6

### Question:

In another pond the amount of pondweed ( $P$ ) grows at a rate proportional to the amount of pondweed already present in the pond. Pondweed is also removed by fish eating it at a constant rate of  $Q$  per unit of time.  
Write down a differential equation relating  $P$  and  $t$ , where  $t$  is the time which has elapsed since the start of the observation.

### Solution:

The rate of increase of pondweed is  $\frac{dP}{dt}$ .

This is proportional to  $P$ .

$$\therefore \frac{dP}{dt} \propto P$$

$$\therefore \frac{dP}{dt} = kP, \text{ where } k \text{ is a constant.}$$

But also pondweed is removed at a rate  $Q$

$$\therefore \frac{dP}{dt} = kP - Q$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise E, Question 7

### Question:

A circular patch of oil on the surface of some water has radius  $r$  and the radius increases over time at a rate inversely proportional to the radius.  
Write down a differential equation relating  $r$  and  $t$ , where  $t$  is the time which has elapsed since the start of the observation.

### Solution:

The rate of increase of the radius is  $\frac{dr}{dt}$ .

$\therefore \frac{dr}{dt} \propto \frac{1}{r}$ , as it is *inversely* proportional to the radius.

$\therefore \frac{dr}{dt} = \frac{k}{r}$ , where  $k$  is the constant of proportion.

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise E, Question 8

#### Question:

A metal bar is heated to a certain temperature, then allowed to cool down and it is noted that, at time  $t$ , the rate of loss of temperature is proportional to the difference in temperature between the metal bar,  $\theta$ , and the temperature of its surroundings  $\theta_0$ .

Write down a differential equation relating  $\theta$  and  $t$ .

#### Solution:

The rate of change of temperature is  $\frac{d\theta}{dt}$ .

$\therefore \frac{d\theta}{dt} \propto - \left( \theta - \theta_0 \right)$  The rate of *loss* indicates the negative sign.

$\therefore \frac{d\theta}{dt} = -k \left( \theta - \theta_0 \right)$ , where  $k$  is the positive constant of proportion.



# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise E, Question 9

### Question:

Fluid flows out of a cylindrical tank with constant cross section. At time  $t$  minutes,  $t > 0$ , the volume of fluid remaining in the tank is  $V \text{ m}^3$ . The rate at which the fluid flows in  $\text{m}^3 \text{ min}^{-1}$  is proportional to the square root of  $V$ . Show that the depth  $h$  metres of fluid in the tank satisfies the differential equation  $\frac{dh}{dt} = -k\sqrt{h}$ , where  $k$  is a positive constant.

### Solution:

Let the rate of flow of fluid be  $-\frac{dV}{dt}$ , as fluid is flowing *out* of the tank, and the volume left in the tank is decreasing.

$$\therefore \frac{-dV}{dt} \propto \sqrt{V}$$

$$\therefore \frac{dV}{dt} = -k'\sqrt{V}, \text{ where } k' \text{ is a positive constant.}$$

But  $V = Ah$ , where  $A$  is the constant cross section.

$$\therefore \frac{dV}{dh} = A$$

Use the chain rule to find  $\frac{dh}{dt}$ :

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = \frac{-k'\sqrt{V}}{A}$$

But  $V = Ah$ ,

$$\therefore \frac{dh}{dt} = \frac{-k'\sqrt{Ah}}{A} = \left( \frac{-k'}{\sqrt{A}} \right) \sqrt{h} = -k\sqrt{h}, \text{ where } \frac{k'}{\sqrt{A}} \text{ is a positive}$$

constant.

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

**Differentiation**  
Exercise E, Question 10

### Question:

At time  $t$  seconds the surface area of a cube is  $A \text{ cm}^2$  and the volume is  $V \text{ cm}^3$ .  
The surface area of the cube is expanding at a constant rate  $2 \text{ cm}^2 \text{ s}^{-1}$ .

Show that  $\frac{dV}{dt} = \frac{1}{2}V^{\frac{1}{3}}$ .

### Solution:

Rate of expansion of surface area is  $\frac{dA}{dt}$ .

Need  $\frac{dV}{dt}$  so use the chain rule.

$$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

$$\text{As } \frac{dA}{dt} = 2, \frac{dV}{dt} = 2 \frac{dV}{dA} \text{ or } 2 \div \left( \frac{dA}{dV} \right) \quad \textcircled{1}$$

Let the cube have edge of length  $x \text{ cm}$ .

Then  $V = x^3$  and  $A = 6x^2$ .

Eliminate  $x$  to give  $A = 6V^{\frac{2}{3}}$

$$\therefore \frac{dA}{dV} = 4V^{-\frac{1}{3}}$$

$$\text{From } \textcircled{1} \quad \frac{dV}{dt} = \frac{2}{4V^{-\frac{1}{3}}} = \frac{2V^{\frac{1}{3}}}{4} = \frac{1}{2}V^{\frac{1}{3}}$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

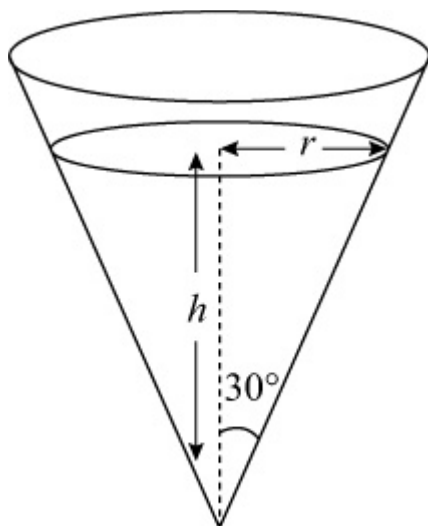
Differentiation  
Exercise E, Question 11

### Question:

An inverted conical funnel is full of salt. The salt is allowed to leave by a small hole in the vertex. It leaves at a constant rate of  $6 \text{ cm}^3 \text{ s}^{-1}$ . Given that the angle of the cone between the slanting edge and the vertical is  $30$  degrees, show that the volume of the salt is  $\frac{1}{9}\pi h^3$ , where  $h$  is the height of salt at time  $t$  seconds.

Show that the rate of change of the height of the salt in the funnel is inversely proportional to  $h^2$ . Write down the differential equation relating  $h$  and  $t$ .

### Solution:



$$\text{Use } V = \frac{1}{3}\pi r^2 h$$

$$\text{As } \tan 30^\circ = \frac{r}{h}$$

$$\therefore r = h \tan 30^\circ = \frac{h}{\sqrt{3}}$$

$$\therefore V = \frac{1}{3}\pi \left( \frac{h^2}{3} \right) \times h = \frac{1}{9}\pi h^3 \quad \text{①}$$

$$\text{It is given that } \frac{dV}{dt} = -6.$$

To find  $\frac{dh}{dt}$  use the chain rule:

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = \frac{dV}{dt} \div \frac{dV}{dh}$$

$$\text{From ① } \frac{dV}{dh} = \frac{1}{3}\pi h^2$$

$$\therefore \frac{dh}{dt} = -6 \div \frac{1}{3}\pi h^2$$

$$\therefore \frac{dh}{dt} = \frac{-18}{\pi h^2}$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise F, Question 1

#### Question:

The curve  $C$  is given by the equations

$$x = 4t - 3, y = \frac{8}{t^2}, t > 0$$

where  $t$  is a parameter.

At  $A$ ,  $t = 2$ . The line  $l$  is the normal to  $C$  at  $A$ .

(a) Find  $\frac{dy}{dx}$  in terms of  $t$ .

(b) Hence find an equation of  $l$ . **(E)**

#### Solution:

$$(a) x = 4t - 3, y = \frac{8}{t^2} = 8t^{-2}$$

$$\therefore \frac{dx}{dt} = 4 \text{ and } \frac{dy}{dt} = -16t^{-3}$$

$$\therefore \frac{dy}{dx} = \frac{-16t^{-3}}{4} = \frac{-4}{t^3}$$

(b) When  $t = 2$  the curve has gradient  $\frac{-4}{8} = -\frac{1}{2}$ .

$\therefore$  the normal has gradient 2.

Also the point  $A$  has coordinates  $(5, 2)$

$\therefore$  the equation of the normal is

$$y - 2 = 2(x - 5)$$

$$\text{i.e. } y = 2x - 8$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 2

### Question:

The curve  $C$  is given by the equations  $x = 2t$ ,  $y = t^2$ , where  $t$  is a parameter. Find an equation of the normal to  $C$  at the point  $P$  on  $C$  where  $t = 3$ . **E**

### Solution:

$$x = 2t, y = t^2$$

$$\frac{dx}{dt} = 2, \frac{dy}{dt} = 2t$$

$$\therefore \frac{dy}{dx} = \frac{2t}{2} = t$$

When  $t = 3$  the gradient of the curve is 3.

$$\therefore \text{the gradient of the normal is } -\frac{1}{3}.$$

Also at the point  $P$  where  $t = 3$ , the coordinates are (6, 9).

$\therefore$  the equation of the normal is

$$y - 9 = -\frac{1}{3} (x - 6)$$

$$\text{i.e. } 3y - 27 = -x + 6$$

$$\therefore 3y + x = 33$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 3

### Question:

The curve  $C$  has parametric equations

$$x = t^3, y = t^2, t > 0$$

Find an equation of the tangent to  $C$  at  $A(1, 1)$ . **E**

### Solution:

$$x = t^3, y = t^2$$

$$\frac{dx}{dt} = 3t^2 \text{ and } \frac{dy}{dt} = 2t$$

$$\therefore \frac{dy}{dx} = \frac{2t}{3t^2} = \frac{2}{3t}$$

At the point  $(1, 1)$  the value of  $t$  is 1.

$\therefore$  the gradient of the curve is  $\frac{2}{3}$ , which is also the gradient of the tangent.

$\therefore$  the equation of the tangent is

$$y - 1 = \frac{2}{3} (x - 1)$$

$$\text{i.e. } y = \frac{2}{3}x + \frac{1}{3}$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise F, Question 4

#### Question:

A curve  $C$  is given by the equations  
 $x = 2 \cos t + \sin 2t$ ,  $y = \cos t - 2 \sin 2t$ ,  $0 < t < \pi$   
 where  $t$  is a parameter.

- (a) Find  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$  in terms of  $t$ .
- (b) Find the value of  $\frac{dy}{dx}$  at the point  $P$  on  $C$  where  $t = \frac{\pi}{4}$ .
- (c) Find an equation of the normal to the curve at  $P$ . **E**

#### Solution:

(a)  $x = 2 \cos t + \sin 2t$ ,  $y = \cos t - 2 \sin 2t$   
 $\frac{dx}{dt} = -2 \sin t + 2 \cos 2t$ ,  $\frac{dy}{dt} = -\sin t - 4 \cos 2t$

(b)  $\therefore \frac{dy}{dx} = \frac{-\sin t - 4 \cos 2t}{-2 \sin t + 2 \cos 2t}$

When  $t = \frac{\pi}{4}$ ,  $\frac{dy}{dx} = \frac{\frac{-1}{\sqrt{2}} - 0}{\frac{-2}{\sqrt{2}} + 0} = \frac{1}{2}$

- (c)  $\therefore$  the gradient of the normal at the point  $P$ , where  $t = \frac{\pi}{4}$ , is  $-2$ .

The coordinates of  $P$  are found by substituting  $t = \frac{\pi}{4}$  into the parametric equations, to give

$$x = \frac{2}{\sqrt{2}} + 1, y = \frac{1}{\sqrt{2}} - 2$$

$\therefore$  the equation of the normal is

$$y - \left( \frac{1}{\sqrt{2}} - 2 \right) = -2 \left[ x - \left( \frac{2}{\sqrt{2}} + 1 \right) \right]$$



$$\text{i.e. } y - \frac{1}{\sqrt{2}} + 2 = -2x + \frac{4}{\sqrt{2}} + 2$$

$$\therefore y + 2x = \frac{5\sqrt{2}}{2}$$

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# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise F, Question 5

#### Question:

A curve is given by  $x = 2t + 3$ ,  $y = t^3 - 4t$ , where  $t$  is a parameter. The point  $A$  has parameter  $t = -1$  and the line  $l$  is the tangent to  $C$  at  $A$ . The line  $l$  also cuts the curve at  $B$ .

(a) Show that an equation for  $l$  is  $2y + x = 7$ .

(b) Find the value of  $t$  at  $B$ . **E**

#### Solution:

(a)  $x = 2t + 3$ ,  $y = t^3 - 4t$

At point  $A$ ,  $t = -1$ .

$\therefore$  the coordinates of the point  $A$  are  $(1, 3)$

$$\frac{dx}{dt} = 2 \text{ and } \frac{dy}{dt} = 3t^2 - 4$$

$$\therefore \frac{dy}{dx} = \frac{3t^2 - 4}{2}$$

At the point  $A$ ,  $\frac{dy}{dx} = -\frac{1}{2}$

$\therefore$  the gradient of the tangent at  $A$  is  $-\frac{1}{2}$ .

$\therefore$  the equation of the tangent at  $A$  is

$$y - 3 = -\frac{1}{2} (x - 1)$$

i.e.  $2y - 6 = -x + 1$

$$\therefore 2y + x = 7$$

(b) This line cuts the curve at the point  $B$ .

$\therefore 2(t^3 - 4t) + (2t + 3) = 7$  gives the values of  $t$  at  $A$  and  $B$ .

i.e.  $2t^3 - 6t - 4 = 0$

At  $A$ ,  $t = -1$

$\therefore (t + 1)$  is a root of this equation

$$2t^3 - 6t - 4 = (t + 1) (2t^2 - 2t - 4) = (t + 1) (t + 1) (2t - 4) = 2(t + 1)^2 (t - 2)$$

So when the line meets the curve,  $t = -1$  (repeated root because the line touches the curve) or  $t = 2$ .

$\therefore$  at the point  $B$ ,  $t = 2$ .

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 6

### Question:

A Pancho car has value £ $V$  at time  $t$  years. A model for  $V$  assumes that the rate of decrease of  $V$  at time  $t$  is proportional to  $V$ . Form an appropriate differential equation for  $V$ . **B**

### Solution:

$\frac{dV}{dt}$  is the rate of change of  $V$ .

$\frac{dV}{dt} \propto -V$ , as a decrease indicates a negative quantity.

$\therefore \frac{dV}{dt} = -kV$ , where  $k$  is a positive constant of proportionality.

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

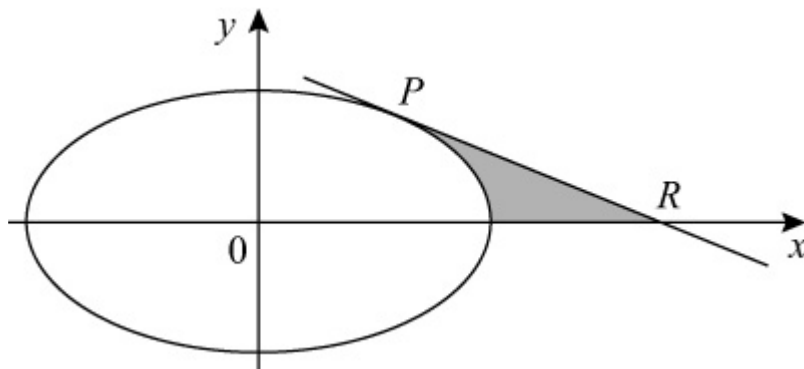
### Differentiation

#### Exercise F, Question 7

#### Question:

The curve shown has parametric equations

$$x = 5 \cos \theta, y = 4 \sin \theta, 0 \leq \theta < 2\pi$$



- (a) Find the gradient of the curve at the point  $P$  at which  $\theta = \frac{\pi}{4}$ .
- (b) Find an equation of the tangent to the curve at the point  $P$ .
- (c) Find the coordinates of the point  $R$  where this tangent meets the  $x$ -axis. **E**

#### Solution:

(a)  $x = 5 \cos \theta, y = 4 \sin \theta$

$$\frac{dx}{d\theta} = -5 \sin \theta \text{ and } \frac{dy}{d\theta} = 4 \cos \theta$$

$$\therefore \frac{dy}{dx} = \frac{-4 \cos \theta}{5 \sin \theta}$$

At the point  $P$ , where  $\theta = \frac{\pi}{4}$ ,  $\frac{dy}{dx} = \frac{-4}{5}$ .

(b) At the point  $P$ ,  $x = \frac{5}{\sqrt{2}}$  and  $y = \frac{4}{\sqrt{2}}$ .

$\therefore$  the equation of the tangent at  $P$  is

$$y - \frac{4}{\sqrt{2}} = \frac{-4}{5} \left( x - \frac{5}{\sqrt{2}} \right)$$

i.e.  $y - \frac{4}{\sqrt{2}} = \frac{-4}{5}x + \frac{4}{\sqrt{2}}$

$$\therefore y = \frac{-4}{5}x + \frac{8}{\sqrt{2}}$$

Multiply equation by 5 and rationalise the denominator of the surd:

$$5y + 4x = 20\sqrt{2}$$

(c) The tangent meets the  $x$ -axis when  $y = 0$ .

$$\therefore x = 5\sqrt{2} \text{ and } R \text{ has coordinates } (5\sqrt{2}, 0).$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise F, Question 8

#### Question:

The curve  $C$  has parametric equations

$$x = 4 \cos 2t, y = 3 \sin t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$A$  is the point  $\left(2, 1\frac{1}{2}\right)$ , and lies on  $C$ .

(a) Find the value of  $t$  at the point  $A$ .

(b) Find  $\frac{dy}{dx}$  in terms of  $t$ .

(c) Show that an equation of the normal to  $C$  at  $A$  is  $6y - 16x + 23 = 0$ .  
The normal at  $A$  cuts  $C$  again at the point  $B$ .

(d) Find the  $y$ -coordinate of the point  $B$ .

#### **E**

#### Solution:

(a)  $x = 4 \cos 2t$  and  $y = 3 \sin t$

$A$  is the point  $\left(2, 1\frac{1}{2}\right)$  and so

$$4 \cos 2t = 2 \text{ and } 3 \sin t = 1\frac{1}{2}$$

$$\therefore \cos 2t = \frac{1}{2} \text{ and } \sin t = \frac{1}{2}$$

As  $-\frac{\pi}{2} < t < \frac{\pi}{2}$ ,  $t = \frac{\pi}{6}$  at the point  $A$ .

(b)  $\frac{dx}{dt} = -8 \sin 2t$  and  $\frac{dy}{dt} = 3 \cos t$

$$\therefore \frac{dy}{dx} = \frac{3 \cos t}{-8 \sin 2t}$$

$$= \frac{-3 \cos t}{16 \sin t \cos t} \quad (\text{using the double angle formula})$$

$$= \frac{-3}{16 \sin t}$$

$$= \frac{-3}{16} \operatorname{cosec} t$$

(c) When  $t = \frac{\pi}{6}$ ,  $\frac{dy}{dx} = \frac{-3}{8}$

$\therefore$  the gradient of the normal at the point  $A$  is  $\frac{8}{3}$ .

$\therefore$  the equation of the normal is

$$y - 1 \frac{1}{2} = \frac{8}{3} (x - 2)$$

Multiply equation by 6:

$$6y - 9 = 16x - 32$$

$$\therefore 6y - 16x + 23 = 0$$

(d) The normal cuts the curve when

$$6(3 \sin t) - 16(4 \cos 2t) + 23 = 0$$

$$\therefore 18 \sin t - 64 \cos 2t + 23 = 0.$$

$$\therefore 18 \sin t - 64(1 - 2 \sin^2 t) + 23 = 0 \quad (\text{using the double angle}$$

formula)

$$\therefore 128 \sin^2 t + 18 \sin t - 41 = 0$$

But  $\sin t = \frac{1}{2}$  is one solution of this equation, as point  $A$  lies on the line and on the curve.

$$\therefore 128 \sin^2 t + 18 \sin t - 41 = (2 \sin t - 1)(64 \sin t + 41)$$

$$\therefore (2 \sin t - 1)(64 \sin t + 41) = 0$$

$$\therefore \text{at point } B, \sin t = \frac{-41}{64}$$

$$\therefore \text{the } y \text{ coordinate of point } B \text{ is } \frac{-123}{64}.$$



# Solutionbank

## Edexcel AS and A Level Modular Mathematics

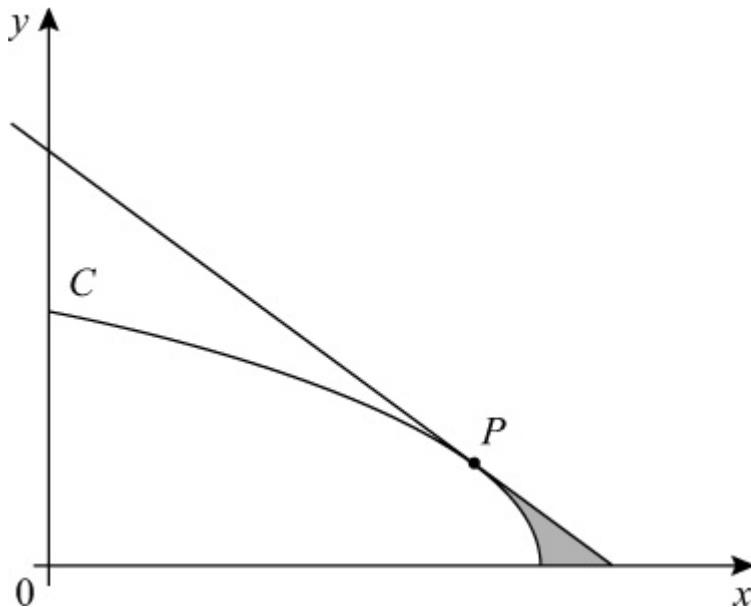
Differentiation  
Exercise F, Question 9

### Question:

The diagram shows the curve  $C$  with parametric equations

$$x = a \sin^2 t, \quad y = a \cos t, \quad 0 \leq t \leq \frac{1}{2}\pi$$

where  $a$  is a positive constant. The point  $P$  lies on  $C$  and has coordinates  $\left( \frac{3}{4}a, \frac{1}{2}a \right)$ .



(a) Find  $\frac{dy}{dx}$ , giving your answer in terms of  $t$ .

(b) Find an equation of the tangent at  $P$  to  $C$ .

**E**

### Solution:

(a)  $x = a \sin^2 t, \quad y = a \cos t$

$$\frac{dx}{dt} = 2a \sin t \cos t \quad \text{and} \quad \frac{dy}{dt} = -a \sin t$$

$$\therefore \frac{dy}{dx} = \frac{-a \sin t}{2a \sin t \cos t} = \frac{-1}{2 \cos t} = \frac{-1}{2} \sec t$$

(b)  $P$  is the point  $\left( \frac{3}{4}a, \frac{1}{2}a \right)$  and lies on the curve.

$$\therefore a \sin^2 t = \frac{3}{4}a \text{ and } a \cos t = \frac{1}{2}a$$

$$\therefore \sin t = \pm \frac{\sqrt{3}}{2} \text{ and } \cos t = \frac{1}{2} \text{ and } 0 \leq t \leq \frac{1}{2}\pi$$

$$\therefore t = \frac{\pi}{3}$$

$\therefore$  the gradient of the curve at point  $P$  is  $-\frac{1}{2} \sec \frac{\pi}{3} = -1$ .

The equation of the tangent at  $P$  is

$$y - \frac{1}{2}a = -1 \left( x - \frac{3}{4}a \right)$$

$$\therefore y + x = \frac{1}{2}a + \frac{3}{4}a$$

Multiply equation by 4 to give  $4y + 4x = 5a$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

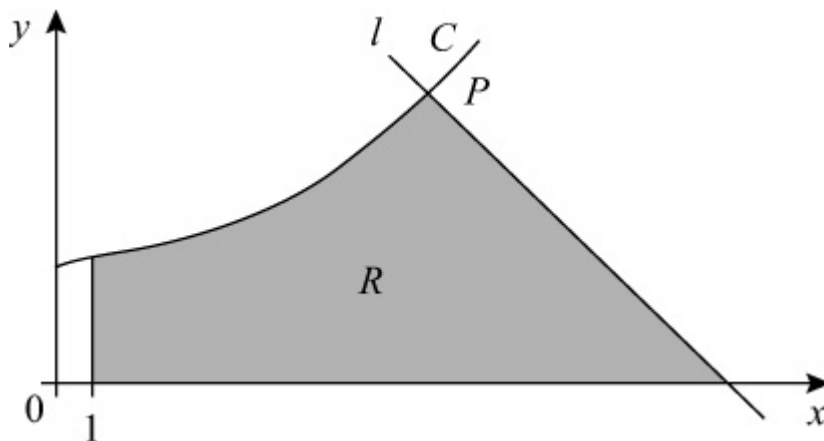
Differentiation  
Exercise F, Question 10

### Question:

This graph shows part of the curve  $C$  with parametric equations

$$x = (t + 1)^2, y = \frac{1}{2}t^3 + 3, t \geq -1$$

$P$  is the point on the curve where  $t = 2$ . The line  $l$  is the normal to  $C$  at  $P$ . Find the equation of  $l$ .



**E**

### Solution:

$$x = (t + 1)^2, y = \frac{1}{2}t^3 + 3$$

$$\frac{dx}{dt} = 2(t + 1) \text{ and } \frac{dy}{dt} = \frac{3}{2}t^2$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{3}{2}t^2\right)}{2(t+1)} = \frac{3t^2}{4(t+1)}$$

$$\text{When } t = 2, \frac{dy}{dx} = \frac{3 \times 4}{4 \times 3} = 1$$

The gradient of the normal at the point  $P$  where  $t = 2$ , is  $-1$ .  
The coordinates of  $P$  are  $(9, 7)$ .

$\therefore$  the equation of the normal is

$$y - 7 = -1(x - 9)$$

$$\text{i.e. } y - 7 = -x + 9$$

$$\therefore y + x = 16$$

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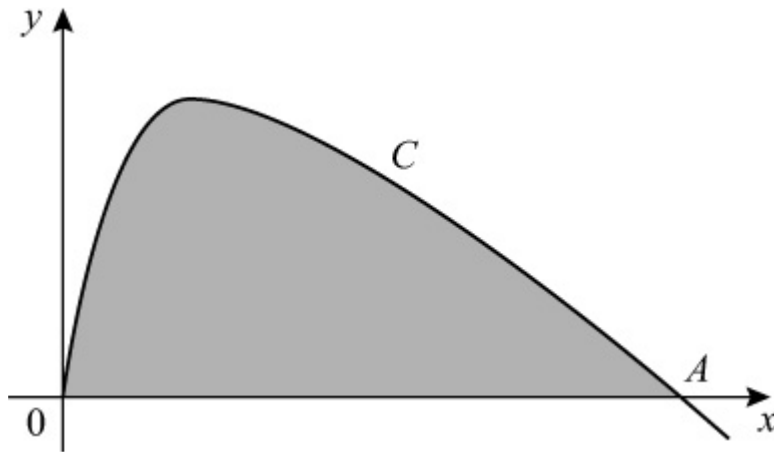
# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 11

### Question:

The diagram shows part of the curve  $C$  with parametric equations  
 $x = t^2$ ,  $y = \sin 2t$ ,  $t \geq 0$   
 The point  $A$  is an intersection of  $C$  with the  $x$ -axis.



- (a) Find, in terms of  $\pi$ , the  $x$ -coordinate of  $A$ .
- (b) Find  $\frac{dy}{dx}$  in terms of  $t$ ,  $t > 0$ .
- (c) Show that an equation of the tangent to  $C$  at  $A$  is  $4x + 2\pi y = \pi^2$ .

**E**

### Solution:

- (a)  $x = t^2$  and  $y = \sin 2t$   
 At the point  $A$ ,  $y = 0$ .  
 $\therefore \sin 2t = 0$   
 $\therefore 2t = \pi$   
 $\therefore t = \frac{\pi}{2}$

The point  $A$  is  $\left( \frac{\pi^2}{4}, 0 \right)$

$$(b) \frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = 2 \cos 2t$$

$$\therefore \frac{dy}{dx} = \frac{\cos 2t}{t}$$

$$(c) \text{ At point A, } \frac{dy}{dx} = \frac{-1}{\left(\frac{\pi}{2}\right)} = \frac{-2}{\pi}$$

$\therefore$  the gradient of the tangent at A is  $\frac{-2}{\pi}$ .

$\therefore$  the equation of the tangent at A is

$$y - 0 = \frac{-2}{\pi} \left( x - \frac{\pi^2}{4} \right)$$

$$\text{i.e. } y = \frac{-2x}{\pi} + \frac{\pi}{2}$$

Multiply equation by  $2\pi$  to give

$$2\pi y + 4x = \pi^2$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 12

### Question:

Find the gradient of the curve with equation  
 $5x^2 + 5y^2 - 6xy = 13$   
at the point (1, 2).

**E**

### Solution:

$$5x^2 + 5y^2 - 6xy = 13$$

Differentiate implicitly with respect to  $x$ :

$$10x + 10y \frac{dy}{dx} - \left( 6x \frac{dy}{dx} + 6y \right) = 0$$

$$\therefore \frac{dy}{dx} \left( 10y - 6x \right) + 10x - 6y = 0$$

At the point (1, 2)

$$\frac{dy}{dx} \left( 14 \right) + 10 - 12 = 0$$

$$\therefore \frac{dy}{dx} = \frac{2}{14} = \frac{1}{7}$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 13

### Question:

Given that  $e^{2x} + e^{2y} = xy$ , find  $\frac{dy}{dx}$  in terms of  $x$  and  $y$ .

**E**

### Solution:

$$e^{2x} + e^{2y} = xy$$

Differentiate with respect to  $x$ :

$$2e^{2x} + 2e^{2y} \frac{dy}{dx} = x \frac{dy}{dx} + y \times 1$$

$$\therefore 2e^{2y} \frac{dy}{dx} - x \frac{dy}{dx} = y - 2e^{2x}$$

$$\therefore \frac{dy}{dx} \left( 2e^{2y} - x \right) = y - 2e^{2x}$$

$$\therefore \frac{dy}{dx} = \frac{y - 2e^{2x}}{2e^{2y} - x}$$

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## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 14

### Question:

Find the coordinates of the turning points on the curve  $y^3 + 3xy^2 - x^3 = 3$ .

**E**

### Solution:

$$y^3 + 3xy^2 - x^3 = 3$$

Differentiate with respect to  $x$ :

$$3y^2 \frac{dy}{dx} + \left( 3x \times 2y \frac{dy}{dx} + y^2 \times 3 \right) - 3x^2 = 0$$

$$\therefore \frac{dy}{dx} \left( 3y^2 + 6xy \right) = 3x^2 - 3y^2$$

$$\therefore \frac{dy}{dx} = \frac{3(x^2 - y^2)}{3y(y + 2x)} = \frac{x^2 - y^2}{y(y + 2x)}$$

When  $\frac{dy}{dx} = 0$ ,  $x^2 = y^2$ , i.e.  $x = \pm y$

$$\text{When } x = +y, y^3 + 3y^3 - y^3 = 3 \Rightarrow 3y^3 = 3 \Rightarrow y = 1 \text{ and } x = 1$$

$$\text{When } x = -y, y^3 - 3y^3 + y^3 = 3 \Rightarrow -y^3 = 3 \Rightarrow y = \sqrt[3]{(-3)} \text{ and } x = -\sqrt[3]{(-3)}$$

$$\therefore \text{ the coordinates are } (1, 1) \text{ and } \left( -\sqrt[3]{(-3)}, \sqrt[3]{(-3)} \right).$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 15

### Question:

Given that  $y(x + y) = 3$ , evaluate  $\frac{dy}{dx}$  when  $y = 1$ .

**E**

### Solution:

$$y(x + y) = 3$$
$$\therefore yx + y^2 = 3$$

Differentiate with respect to  $x$ :

$$\left( y + x \frac{dy}{dx} \right) + 2y \frac{dy}{dx} = 0 \quad \textcircled{1}$$

When  $y = 1$ ,  $1(x + 1) = 3$  (from original equation)

$$\therefore x = 2$$

Substitute into  $\textcircled{1}$ :

$$1 + 2 \frac{dy}{dx} + 2 \frac{dy}{dx} = 0$$

$$\therefore 4 \frac{dy}{dx} = -1$$

$$\text{i.e. } \frac{dy}{dx} = \frac{-1}{4}$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 16

### Question:

- (a) If  $(1 + x)(2 + y) = x^2 + y^2$ , find  $\frac{dy}{dx}$  in terms of  $x$  and  $y$ .
- (b) Find the gradient of the curve  $(1 + x)(2 + y) = x^2 + y^2$  at each of the two points where the curve meets the  $y$ -axis.
- (c) Show also that there are two points at which the tangents to this curve are parallel to the  $y$ -axis.

**E**

### Solution:

(a)  $(1 + x)(2 + y) = x^2 + y^2$

Differentiate with respect to  $x$ :

$$\left(1 + x\right) \left(\frac{dy}{dx}\right) + \left(2 + y\right) \left(1\right) = 2x + 2y \frac{dy}{dx}$$

$$\therefore \left(1 + x - 2y\right) \frac{dy}{dx} = 2x - y - 2$$

$$\therefore \frac{dy}{dx} = \frac{2x - y - 2}{1 + x - 2y}$$

- (b) When the curve meets the  $y$ -axis,  $x = 0$ .

Put  $x = 0$  in original equation  $(1 + x)(2 + y) = x^2 + y^2$ .

Then  $2 + y = y^2$

i.e.  $y^2 - y - 2 = 0$

$$\Rightarrow (y - 2)(y + 1) = 0$$

$$\therefore y = 2 \text{ or } y = -1 \text{ when } x = 0$$

At  $(0, 2)$ ,  $\frac{dy}{dx} = \frac{-4}{-3} = \frac{4}{3}$

At  $(0, -1)$ ,  $\frac{dy}{dx} = \frac{-1}{3}$

- (c) When the tangent is parallel to the  $y$ -axis it has infinite gradient and as

$$\frac{dy}{dx} = \frac{2x - y - 2}{1 + x - 2y}$$

So  $1 + x - 2y = 0$

Substitute  $1 + x = 2y$  into the equation of the curve:

$$2y(2 + y) = (2y - 1)^2 + y^2$$

$$2y^2 + 4y = 4y^2 - 4y + 1 + y^2$$

$$3y^2 - 8y + 1 = 0$$

$$y = \frac{8 \pm \sqrt{64 - 12}}{6} = \frac{4 \pm \sqrt{13}}{3}$$

When  $y = \frac{4 + \sqrt{13}}{3}$ ,  $x = \frac{5 + 2\sqrt{13}}{3}$

When  $y = \frac{4 - \sqrt{13}}{3}$ ,  $x = \frac{5 - 2\sqrt{13}}{3}$

$\therefore$  there are two points at which the tangents are parallel to the  $y$ -axis.

They are  $\left( \frac{5 + 2\sqrt{13}}{3}, \frac{4 + \sqrt{13}}{3} \right)$  and  $\left( \frac{5 - 2\sqrt{13}}{3}, \frac{4 - \sqrt{13}}{3} \right)$ .

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 17

### Question:

A curve has equation  $7x^2 + 48xy - 7y^2 + 75 = 0$ .  $A$  and  $B$  are two distinct points on the curve and at each of these points the gradient of the curve is equal to  $\frac{2}{11}$ . Use implicit differentiation to show that  $x + 2y = 0$  at the points  $A$  and  $B$ .

**E**

### Solution:

$$7x^2 + 48xy - 7y^2 + 75 = 0$$

Differentiate with respect to  $x$  (implicit differentiation):

$$14x + \left( 48x \frac{dy}{dx} + 48y \right) - 14y \frac{dy}{dx} = 0$$

Given that  $\frac{dy}{dx} = \frac{2}{11}$

$$\therefore 14x + 48x \times \frac{2}{11} + 48y - 14y \times \frac{2}{11} = 0$$

Multiply equation by 11,  
then  $154x + 96x + 528y - 28y = 0$

$$\therefore 250x + 500y = 0$$

i.e.  $x + 2y = 0$ , after division by 250.

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 18

### Question:

Given that  $y = x^x$ ,  $x > 0$ ,  $y > 0$ , by taking logarithms show that

$$\frac{dy}{dx} = x^x \left( 1 + \ln x \right)$$

**E**

### Solution:

$$y = x^x$$

Take natural logs of both sides:

$$\ln y = \ln x^x$$

$$\therefore \ln y = x \ln x \quad \text{Property of lns}$$

Differentiate with respect to  $x$ :

$$\frac{1}{y} \frac{dy}{dx} = x \times \frac{1}{x} + \ln x \times 1$$

$$\frac{1}{y} \frac{dy}{dx} = 1 + \ln x$$

$$\therefore \frac{dy}{dx} = y \left( 1 + \ln x \right)$$

But  $y = x^x$

$$\therefore \frac{dy}{dx} = x^x \left( 1 + \ln x \right)$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise F, Question 19

#### Question:

(a) Given that  $x = 2^t$ , by using logarithms prove that

$$\frac{dx}{dt} = 2^t \ln 2$$

A curve  $C$  has parametric equations  $x = 2^t$ ,  $y = 3t^2$ . The tangent to  $C$  at the point with coordinates  $(2, 3)$  cuts the  $x$ -axis at the point  $P$ .

(b) Find  $\frac{dy}{dx}$  in terms of  $t$ .

(c) Calculate the  $x$ -coordinate of  $P$ , giving your answer to 3 decimal places.

#### **E**

#### Solution:

(a) Given  $x = 2^t$

Take natural logs of both sides:

$$\ln x = \ln 2^t = t \ln 2$$

Differentiate with respect to  $t$ :

$$\frac{1}{x} \frac{dx}{dt} = \ln 2$$

$$\therefore \frac{dx}{dt} = x \ln 2 = 2^t \ln 2$$

(b)  $x = 2^t$ ,  $y = 3t^2$

$$\frac{dx}{dt} = 2^t \ln 2, \quad \frac{dy}{dt} = 6t$$

$$\therefore \frac{dy}{dx} = \frac{6t}{2^t \ln 2}$$

(c) At the point  $(2, 3)$ ,  $t = 1$ .

The gradient of the curve at  $(2, 3)$  is  $\frac{6}{2 \ln 2}$ .

$\therefore$  the equation of the tangent is

$$y - 3 = \frac{6}{2 \ln 2} (x - 2)$$

$$\text{i.e. } y = \frac{3}{\ln 2}x - \frac{6}{\ln 2} + 3$$

The tangent meets the  $x$ -axis when  $y = 0$ .

$$\therefore \frac{3}{\ln 2}x = \frac{6}{\ln 2} - 3$$

$$\therefore x = 2 - \ln 2 = 1.307 \text{ (3 decimal places)}$$



# Solutionbank

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise F, Question 20

#### Question:

- (a) Given that  $a^x \equiv e^{kx}$ , where  $a$  and  $k$  are constants,  $a > 0$  and  $x \in \mathbb{R}$ , prove that  $k = \ln a$ .
- (b) Hence, using the derivative of  $e^{kx}$ , prove that when  $y = 2^x$
- $$\frac{dy}{dx} = 2^x \ln 2.$$
- (c) Hence deduce that the gradient of the curve with equation  $y = 2^x$  at the point  $(2, 4)$  is  $\ln 16$ .

#### **E**

#### Solution:

(a)  $a^x = e^{kx}$

Take lns of both sides:

$$\ln a^x = \ln e^{kx}$$

$$\text{i.e. } x \ln a = kx$$

As this is true for all values of  $x$ ,  $k = \ln a$ .

(b) Therefore,  $2^x = e^{\ln 2 \times x}$

$$\text{When } y = 2^x = e^{\ln 2 \times x}$$

$$\frac{dy}{dx} = \ln 2 e^{\ln 2 \times x} = \ln 2 \times 2^x$$

(c) At the point  $(2, 4)$ ,  $x = 2$ .

$\therefore$  the gradient of the curve is

$$\begin{aligned} & 2^2 \ln 2 \\ & = 4 \ln 2 \\ & = \ln 2^4 \quad (\text{property of logs}) \\ & = \ln 16 \end{aligned}$$

# Solutionbank

## Edexcel AS and A Level Modular Mathematics

Differentiation  
Exercise F, Question 21

### Question:

A population  $P$  is growing at the rate of 9% each year and at time  $t$  years may be approximated by the formula

$$P = P_0 (1.09)^t, t \geq 0$$

where  $P$  is regarded as a continuous function of  $t$  and  $P_0$  is the starting population at time  $t = 0$ .

(a) Find an expression for  $t$  in terms of  $P$  and  $P_0$ .

(b) Find the time  $T$  years when the population has doubled from its value at  $t = 0$ ,  
giving your answer to 3 significant figures.

(c) Find, as a multiple of  $P_0$ , the rate of change of population  $\frac{dP}{dt}$  at time  $t = T$ . **E**

### Solution:

(a)  $P = P_0 (1.09)^t$

Take natural logs of both sides:

$$\ln P = \ln [P_0 (1.09)^t] = \ln P_0 + t \ln 1.09$$

$$\therefore t \ln 1.09 = \ln P - \ln P_0$$

$$\Rightarrow t = \frac{\ln P - \ln P_0}{\ln 1.09} \quad \text{or} \quad \frac{\ln \left( \frac{P}{P_0} \right)}{\ln 1.09}$$

(b) When  $P = 2P_0$ ,  $t = T$ .

$$\therefore T = \frac{\ln 2}{\ln 1.09} = 8.04 \text{ (to 3 significant figures)}$$

(c)  $\frac{dP}{dt} = P_0 (1.09)^t \ln 1.09$

When  $t = T$ ,  $P = 2P_0$  so  $(1.09)^T = 2$  and

$$\begin{aligned}\frac{dP}{dt} &= P_0 \times 2 \times \ln 1.09 \\ &= \ln ( 1.09^2 ) \times P_0 = \ln ( 1.1881 ) \times P_0 \\ &= 0.172P_0 \text{ (to 3 significant figures)}\end{aligned}$$

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